TREE-LIKE CONSTRUCTIONS IN TOPOLOGY AND MODAL LOGIC

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Abstract. Within ZFC, we develop a general technique to topologize trees that provides a uniform approach to topological completeness results in modal logic with respect to zero-dimensional Hausdorff spaces. Embeddings of these spaces into well-known extremally disconnected spaces then gives new completeness results for logics extending $\mathsf{S4.2}$.

1. Introduction

Topological semantics of modal logic has a long history. It was shown by McKinsey and Tarski [22] that if we interpret $\Box$ as interior and hence $\Diamond$ as closure, then $\mathsf{S4}$ is the modal logic of all topological spaces. Many topological completeness results have been obtained since the inception of topological semantics. Below we give a short list for the logics that play an important role in the paper.

- $\mathsf{S4}$ is the logic of any crowded metric space [22, 25]. This result is often referred to as the *McKinsey-Tarski theorem*.
- Grz is the logic of any ordinal space $\alpha \geq \omega^\omega$ [1, 14].
- Grz_n (for nonzero $n \in \omega$) is the logic of any ordinal space α satisfying $\omega^{n-1} + 1 \leq \alpha \leq \omega^n$ [1] (see also [12, Sec. 6]).
- $\mathsf{S4.1}$ is the logic of the Pełczyński compactification of the discrete space ω (that is, the compactification of ω whose remainder is homeomorphic to the Cantor space) [11, Cor. 3.19].

If in the second bullet we restrict to a countable α, then all the above completeness results concern metric spaces. In fact, as was shown in [7], the logics above are the only ones arising as the logic of a metric space.

It is a consequence of the McKinsey-Tarski theorem that $\mathsf{S4}$ is the logic of the Cantor space. An alternative proof of this result was given in [23] (see also [2]), where the infinite binary tree was utilized. Kremer [21] used the infinite binary tree with limits to prove that $\mathsf{S4}$ is strongly complete for any crowded metric space. Further utility of trees with limits is demonstrated in [8].

Herein we develop a general technique of topologizing trees which allows us to provide a uniform approach to topological completeness results for zero-dimensional Hausdorff spaces. It also allows us to obtain new topological completeness results with respect to non-metrizable spaces. Embedding these spaces into well-known extremally disconnected spaces (ED-spaces for short) then yields new completeness results for the logics above $\mathsf{S4.2}$ indicated in Figure 1.

It was proved in [10] that $\mathsf{S4.1.2}$ is the logic of the Čech-Stone compactification $\beta\omega$ of the discrete space ω, and this result was utilized in [11] to show that $\mathsf{S4.2}$ is the logic of the Gleason cover of the real unit interval $[0, 1]$. However, these results require a set-theoretic

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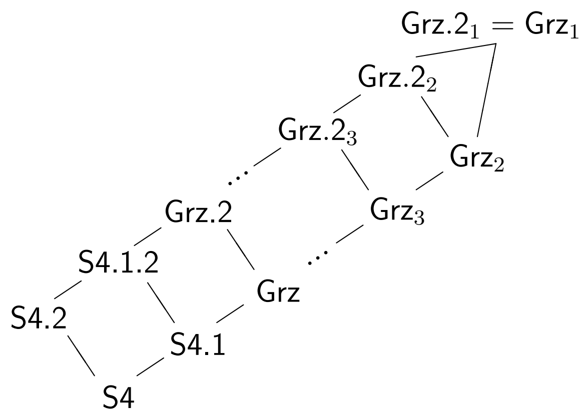

FIGURE 1. Some well-known extensions of $\mathsf{S4}$.

axiom beyond ZFC, and it remains an open problem whether these results are true in ZFC. In contrast, all our results are obtained within ZFC.

We briefly outline some of the techniques employed to obtain the indicated completeness results. A unified way of obtaining a zero-dimensional topology on an infinite tree with limits, say T, is by designating a particular Boolean algebra of subsets of T as a basis. If T has countable branching, then the topology ends up being metrizable. If the branching is 1, then the obtained space is homeomorphic to the ordinal space $\omega + 1$; if the branching is ≥ 2 but finite, then it is homeomorphic to the Pełczyński compactification of ω; and if the branching is countably infinite, then there are subspaces homeomorphic to the space of rational numbers, the Baire space, as well as to the ordinal spaces $\omega^n + 1$. The latter subspaces can be thought of as being recursively built from $\omega + 1$, which is (homeomorphic to) the one-point compactification of ω.

For uncountable branching, it is required to designate a Boolean σ-algebra as a basis for the topology. This leads to topological completeness results for $\mathsf{S4}, \mathsf{S4.1}, \mathsf{Grz}$, and Grz_n with respect to non-metrizable zero-dimensional Hausdorff spaces. This increase in cardinality results in the one-point compactification of ω being replaced by the one-point Lindelöfication of an uncountable discrete space.

To obtain topological completeness results for logics extending $\mathsf{S4.2}$, we select a dense subspace of either the Čech-Stone compactification βD of a discrete space D with large cardinality or the Gleason cover E of a large enough power of $[0, 1]$. This selection is realized by embedding a subspace of an uncountable branching tree with limits into either βD or E. The latter gives rise to $\mathsf{S4.2}$, while the former yields the other logics of interest extending $\mathsf{S4.2}$. We point out that these constructions can be done in ZFC.

The paper is organized as follows. In Section 2 we recall pertinent definitions and results from both modal logic and topology. In Section 3 we introduce the main object of study, trees with limits, and define multiple topologies on such trees. Section 4 proves some mapping theorems for countable branching trees with limits, which lead to alternate proofs of some well-known topological completeness results with respect to zero-dimensional metrizable spaces in Section 5. In Section 6 we generalize the patch topology on trees with limits to the σ-patch topology. Section 7 calculates the logics of certain subspaces associated with the aforementioned σ-patch topology, yielding topological completeness results for non-metrizable zero-dimensional Hausdorff spaces. The paper concludes with Section 8, which proves new topological completeness results for extremally disconnected Tychonoff spaces. These results are obtained by embedding some of the spaces defined in Section 6 into either the Čech-Stone compactification of a (large) discrete space or the Gleason cover of a (large) power of the closed real unit interval.

2. Background

We assume the reader's familiarity with modal logic and topology. We use [16, 13] as our main references for modal logic, and [18, 20] as our main references for topology.

2.1. **Some basic modal logic.** The logic $\mathsf{S4}$ is the least set of formulas of the basic modal language containing the classical tautologies, the axioms $\Box(p \to q) \to (\Box p \to \Box q)$, $\Box p \to p$, $\Box p \to \Box\Box p$, and closed under the inference rules of modus ponens (MP) $\frac{\varphi,\ \varphi \to \psi}{\psi}$, substitution (S) $\frac{\varphi(p_1,\dots,p_n)}{\varphi(\psi_1,\dots,\psi_n)}$, and necessitation (N) $\frac{\varphi}{\Box\varphi}$.

For a modal formula φ and a modal logic L, we denote by $\mathsf{L} + \varphi$ the logic realized as the least set of formulas containing L and φ, and closed under MP, S, and N. The modal formulas in Table 1 are used to define the modal logics of interest appearing in Table 2. As usual $\Diamond\varphi$ is an abbreviation for $\neg\Box\neg\varphi$.

Notation	Formula
ma	$\Box\Diamond p \to \Diamond\Box p$
ga	$\Diamond\Box p \to \Box\Diamond p$
grz	$\Box(\Box(p \to \Box p) \to p) \to p$
bd_1	$\Diamond\Box p_1 \to p_1$
bd_{n+1}	$\Diamond(\Box p_{n+1} \wedge \neg\mathsf{bd}_n) \to p_{n+1}$ for $n \geq 1$

Table 1. Formulas of interest.

Logic	Axiomatization
$\mathsf{S4.1}$	$\mathsf{S4} + \mathsf{ma}$
$\mathsf{S4.2}$	$\mathsf{S4} + \mathsf{ga}$
$\mathsf{S4.1.2}$	$\mathsf{S4} + \mathsf{ma} + \mathsf{ga}$
Grz	$\mathsf{S4} + \mathsf{grz}$
$\mathsf{Grz.2}$	$\mathsf{Grz} + \mathsf{ga}$
Grz_n	$\mathsf{Grz} + \mathsf{bd}_n$ for $n \geq 1$
$\mathsf{Grz.2}_n$	$\mathsf{Grz.2} + \mathsf{bd}_n$ for $n \geq 1$

Table 2. Logics of interest.

It is well known that every logic in the list has the finite model property (FMP), and hence is complete with respect to its finite Kripke frames. We recall that an $\mathsf{S4}$-frame is a tuple $\mathfrak{F} = (W, R)$ where W is a nonempty set and R is a reflexive and transitive binary relation on W. For $w \in W$, let $R(w) = \{v \in W \mid wRv\}$ and $R^{-1}(w) = \{v \in W \mid vRw\}$. When R is additionally antisymmetric, and hence $\mathfrak{F}$ is a poset, we may write $\leq$ for R, ${\uparrow}w$ for $R(w)$, and ${\downarrow}w$ for $R^{-1}(w)$.

As usual, call $r \in W$ a *root* of $\mathfrak{F}$ if $R(r) = W$, and say that $\mathfrak{F}$ is *rooted* if it has a root. A *quasi-chain* in $\mathfrak{F}$ is $C \subseteq W$ such that for any $w, v \in C$, either wRv or vRw. A *chain* is a quasi-chain that is partially ordered by (the restriction of) R. A finite rooted $\mathsf{S4}$-frame $\mathfrak{F}$ is a *quasi-tree* provided for each $w \in W$, we have that $R^{-1}(w)$ is a quasi-chain. A finite quasi-tree is a *tree* if it is a poset.

Call $w \in W$ *quasi-maximal* (resp. *maximal*) in $\mathfrak{F}$ provided wRv implies vRw (resp. $v = w$). Let $\mathrm{qmax}(\mathfrak{F})$ (resp. $\max(\mathfrak{F})$) denote the set of quasi-maximal (resp. maximal) points of $\mathfrak{F}$.

A *cluster* of $\mathfrak{F}$ is an equivalence class of the equivalence relation given by $w \sim v$ iff wRv and vRw. The relation R induces a partial order on the set of clusters and a *maximal cluster* is a maximal element in this poset.

The *depth* of $\mathfrak{F}$ is n provided there is a chain in $\mathfrak{F}$ consisting of n elements and each chain in $\mathfrak{F}$ consists of at most n elements. A *top-thin-quasi-tree* is a quasi-tree $\mathfrak{F}$ constructed from a given finite quasi-tree $\mathfrak{G}$ by inserting a new maximal point above each maximal cluster in $\mathfrak{G}$, as indicated in Figure 2.

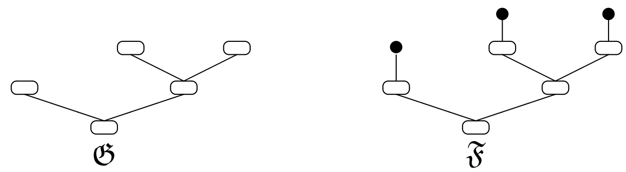

FIGURE 2. Top-thin-quasi-tree $\mathfrak{F}$ obtained from $\mathfrak{G}$.

For $\mathsf{S4}$-frames $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (V, Q)$, a *p-morphism* is a function $f : W \to V$ such that $f^{-1}(Q^{-1}(v)) = R^{-1}(f^{-1}(v))$ for each $v \in V$. An onto p-morphism preserves validity, and hence reflects refutations, meaning that $\mathfrak{F}$ refutes a formula φ whenever $\mathfrak{G}$ does (see, e.g., [16, Thm. 3.15]). Table 3 gathers together some well-known Kripke completeness results for the logics of interest.

Logic	is complete with respect to
$\mathsf{S4}$	finite quasi-trees
$\mathsf{S4.1}$	finite top-thin-quasi-trees
Grz	finite trees
Grz_n	finite trees of depth $\leq n$
$\mathsf{S4.2}$	finite rooted $\mathsf{S4}$-frames with a unique maximal cluster C such that the subframe $W \setminus C$ is a quasi-tree
$\mathsf{S4.1.2}$	finite rooted $\mathsf{S4}$-frames with a unique maximal point m such that the subframe $W \setminus \{m\}$ is a quasi-tree
$\mathsf{Grz.2}$	finite rooted posets with a unique maximal point m such that the subframe $W \setminus \{m\}$ is a tree
$\mathsf{Grz.2}_n$	finite rooted posets of depth $\leq n$ with a unique maximal point m such that the subframe $W \setminus \{m\}$ is a tree

TABLE 3. Kripke completeness for logics of interest.

2.2. **Some basic topology.** In topological semantics, the modal language is interpreted in a topological space X by evaluating propositional variables as subsets of X, classical connectives as Boolean operations on the powerset $\wp(X)$, $\Box$ as the interior operator, and hence $\Diamond$ as the closure operator. A formula φ is *valid* in X, denoted $X \vDash \varphi$, provided φ evaluates to X under any evaluation of the propositional variables. The *logic* of X is $\mathsf{Log}(X) := \{\varphi \mid X \vDash \varphi\}$, and we have $\mathsf{S4} \subseteq \mathsf{Log}(X)$.

Topological semantics generalizes Kripke semantics for $\mathsf{S4}$ as follows. Given an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$, sets of the form $R(w)$ for $w \in W$ form a basis for the topology on W known as the *Alexandroff topology* in which the closure operator is given by R^{-1}. A formula is valid in $\mathfrak{F}$ iff it is valid in the corresponding Alexandroff space, so Kripke completeness immediately transfers to topological completeness. We therefore identify an $\mathsf{S4}$-frame with its corresponding Alexandroff space. But Alexandroff spaces do not satisfy strong separation axioms (unless they are discrete). Thus, it is a nontrivial matter to seek out completeness results for spaces that satisfy stronger separation axioms such as Tychonoff spaces.

We next recall some well-known definitions. A map $f : X \to Y$ between topological spaces is *continuous* if the inverse image of each open in Y is open in X, it is *open* if the image of each open in X is open in Y, and it is *interior* if it is both continuous and open. Interior maps between topological spaces generalize p-morphisms between $\mathsf{S4}$-frames, and they share the following important feature with p-morphisms: onto interior maps reflect refutations.

A topological space X is *zero-dimensional* if it has a basis of clopen (closed and open) sets, and X is *extremally disconnected* (ED) if the closure of each open set is open. It is easy to see that every regular ED-space is zero-dimensional.

For a space X and $A \subseteq X$, let $\mathsf{d}(A)$ denote the set of limit points of A. By transfinite recursion we define $\mathsf{d}^0(A) = A$, $\mathsf{d}^{\alpha+1}(A) = \mathsf{d}(\mathsf{d}^\alpha(A))$, and $\mathsf{d}^\alpha(A) = \bigcap_{\beta<\alpha} \mathsf{d}^\beta(A)$ if α is a limit ordinal. By the Cantor-Bendixson theorem (see, e.g., [27, Thm. 8.5.2]), there is an ordinal α such that $\mathsf{d}^{\alpha+1}(X) = \mathsf{d}^\alpha(X)$, and the least such ordinal is the *Cantor-Bendixson rank* of X, denoted herein by $r(X)$.

Let $\mathsf{Iso}(X)$ denote the isolated points of X. Then $\mathsf{Iso}(X) = X \setminus \mathsf{d}(X)$. Call X *crowded* or *dense-in-itself* if $\mathsf{Iso}(X) = \varnothing$; equivalently, in terms of the Cantor-Bendixson rank, if $r(X) = 0$. Call X *scattered* provided that X contains no nonempty crowded subspace; equivalently, there is an ordinal α such that $\mathsf{d}^\alpha(X) = \varnothing$. Following a suggestion by A. V. Arhangel'skii, we call X *densely discrete* if $\mathsf{Iso}(X)$ is dense in X.[1] It is routine to check that a scattered space is densely discrete, but that the converse is not true in general. Table 4 gathers together some well-known topological completeness results for the logics of interest.

Logic	is complete with respect to
$\mathsf{S4}$	the class of topological spaces
$\mathsf{S4.1}$	the class of densely discrete spaces
Grz	the class of scattered spaces
Grz_n	the class of scattered spaces of Cantor-Bendixson rank $\leq n$
$\mathsf{S4.2}$	the class of ED-spaces
$\mathsf{S4.1.2}$	the class of densely discrete ED-spaces
$\mathsf{Grz.2}$	the class of scattered ED-spaces
$\mathsf{Grz.2}_n$	the class of scattered ED-spaces of Cantor-Bendixson rank $\leq n$

TABLE 4. Topological completeness for logics of interest.

For a scattered Hausdorff space, [5, Thm. 4.9] demonstrates that finite Cantor-Bendixson rank is characterized by the concept of modal Krull dimension—a topological analogue of the depth of an $\mathsf{S4}$-frame—introduced in [4]. For a topological analogue of cluster size, we recall that a space X is *resolvable* provided it contains a dense subset whose complement is also dense. A space is *irresolvable* if it is not resolvable. For nonzero $n \in \omega$, X is *n-resolvable* provided there is a partition of X consisting of n dense subsets. By [4, Lem. 5.9], a space X is n-resolvable iff an n-element cluster is an interior image of X.

We next recall that a closed subset F of X is *irreducible* if it is a join-irreducible element in the lattice of closed subsets of X, and X is a *sober space* provided each irreducible closed subset of X is the closure of a unique singleton. The space X is *coherent* if the set of compact

[1] The notion of densely discrete has appeared in the literature under the names α-scattered [26] and weakly scattered (see, e.g., [6, 11, 7]). Since the term weakly scattered is often used for a different concept (see, e.g., [24, p. 120]), we adopt Arhangel'skii's more descriptive terminology.

open subsets of X forms a basis that is a bounded sublattice of the lattice of open subsets of X. For a poset $(X, \preceq)$, we call $U \subseteq X$ an $\preceq$-*upset* provided $x \in U$ and $x \preceq y$ imply $y \in U$.

Definition 2.1.
(1) A topological space is a *spectral space* if it is sober and coherent.
(2) A topological space is a *Stone space* if it is compact Hausdorff zero-dimensional.
(3) A *Priestley space* is a tuple $(X, \preceq)$ where X is a Stone space, $\preceq$ is a partial order on X, and $x \not\preceq y$ implies that there is a clopen $\preceq$-upset U such that $x \in U$ and $y \notin U$.

It is well known that spectral spaces and Priestley spaces are closely related to each other. If $(X, \preceq)$ is a Priestley space, then the set of open $\preceq$-upsets is a spectral topology on X in which the compact opens are exactly the clopen $\preceq$-upsets. Conversely, let (X, τ) be a spectral space and let c_τ be the closure in (X, τ). Define the *patch topology* π to be the topology generated by the compact opens of (X, τ) and their complements. Then $(X, \pi, \preceq)$ is a Priestley space, where $\preceq$ is the *specialization order* of (X, τ) given by $x \preceq y$ iff $x \in \mathsf{c}_\tau\{y\}$. Moreover, the clopen $\preceq$-upsets of $(X, \pi, \preceq)$ are exactly the compact opens of (X, τ). From this it follows that there is a 1-1 correspondence between spectral spaces and Priestley spaces.

Definition 2.2.
(1) A regular space X is *Lindelöf* provided every open cover has a countable subcover.
(2) A Tychonoff space X is a *P-space* provided that each G_δ-set (countable intersection of open sets) is open.

It is well known (see, e.g., [19, pp. 62–63]) that any P-space is zero-dimensional, and that the Boolean algebra of clopens is a σ-algebra.

3. Topologies associated with κ-branching trees

We view cardinals as initial ordinals. Let κ be a nonzero cardinal number. We will be interested in three cases: κ is finite and nonzero ($0 \neq \kappa < \omega$), κ is countably infinite ($\kappa = \omega$), and κ is uncountable ($\kappa \geq \omega_1$).

A *sequence* in κ is a function $\sigma : \alpha \to \kappa$ for some $\alpha \leq \omega$. Let T_κ denote the set of sequences in κ. Call $\sigma \in T_\kappa$ *finite* provided $\alpha < \omega$; otherwise call σ *infinite*. If σ is finite, then we say that the *length* of σ is α, and write $\ell(\sigma) = \alpha$. If σ is infinite, then we say that the *length* of σ is infinite, and write $\ell(\sigma) = \infty$. Note that there is a unique element of T_κ of length 0, namely the *empty* sequence $\varepsilon : \varnothing \to \kappa$.

For sequences $\sigma : \alpha \to \kappa$ and $\varsigma : \beta \to \kappa$, we say that σ is an *initial segment* of ς provided $\alpha \leq \beta$ and $\alpha(n) = \beta(n)$ for all $n < \alpha$; in such case, we write $\sigma \leq \varsigma$. It is routine to check that $\leq$ is a partial ordering of T_κ. For any $\sigma \in T_\kappa$, let ${\uparrow}\sigma = \{\varsigma \in T_\kappa \mid \sigma \leq \varsigma\}$ and ${\downarrow}\sigma = \{\varsigma \in T_\kappa \mid \varsigma \leq \sigma\}$. Note that ${\downarrow}\sigma$ is a chain for each $\sigma \in T_\kappa$; that is, either $\varsigma \leq \varsigma'$ or $\varsigma' \leq \varsigma$ for any $\varsigma, \varsigma' \in {\downarrow}\sigma$.

For $n \in \omega$, if $\sigma : n \to \kappa$ is an initial segment of $\varsigma : n+1 \to \kappa$, we say that σ is the *parent* of ς and that ς is a *child* of σ, and write $\varsigma = \sigma.\varsigma(n)$. For an infinite sequence $\sigma : \omega \to \kappa$, we write $\sigma|_n$ to denote the restriction of σ to n. Thus, $\sigma|_n$ is an initial segment of σ of length n.

For $n \in \omega$, let $T_\kappa^n = \{\sigma \in T_\kappa \mid \ell(\sigma) \leq n\}$ be the set of finite sequences of length at most n, let $T_\kappa^\omega = \{\sigma \in T_\kappa \mid \ell(\sigma) < \omega\}$ be the set of all finite sequences, and let $T_\kappa^\infty = T_\kappa \setminus T_\kappa^\omega$ be the set of all infinite sequences.

Definition 3.1. We let $\mathcal{T}_\kappa = (T_\kappa, \leq)$, $\mathcal{T}_\kappa^n = (T_\kappa^n, \leq)$, and $\mathcal{T}_\kappa^\omega = (T_\kappa^\omega, \leq)$.

Definition 3.2. Let $\mathbf{T}_\kappa = (T_\kappa, \tau)$ be the topological space where τ is generated by the basis $\mathscr{S} := \{{\uparrow}\sigma \mid \sigma \in T_\kappa^\omega\}$.

Remark 3.3. We use normal font for denoting sets, calligraphic font for denoting trees, and boldface font for denoting the topological space whose topology is generated by $\mathscr{S}$.

To see that $\mathbf{T}_\kappa$ is a spectral space, we require the following lemma.

Lemma 3.4. *The compact opens of $\mathbf{T}_\kappa$ are exactly the finite unions of members of $\mathscr{S}$.*

Proof. Clearly each member of $\mathscr{S}$ is compact open, and hence so is a finite union of members of $\mathscr{S}$. Suppose $U \subseteq T_\kappa$ is compact open. Because U is open in $\mathbf{T}_\kappa$, for each $\sigma \in U$, there is $U_\sigma \in \mathscr{S}$ such that $\sigma \in U_\sigma \subseteq U$. Therefore, $\mathscr{C} := \{U_\sigma \mid \sigma \in U\}$ is an open cover of U. As U is compact, there is a finite subcover $\mathscr{C}_0$ of $\mathscr{C}$. Thus, $U = \bigcup \mathscr{C}_0$, as desired. $\square$

Theorem 3.5. *The space $\mathbf{T}_\kappa$ is a spectral space.*

Proof. Let $\sigma, \varsigma \in T_\kappa^\omega$. If $\sigma' \in {\uparrow}\sigma \cap {\uparrow}\varsigma$, then since $\sigma, \varsigma \in {\downarrow}\sigma'$, either $\sigma \leq \varsigma$ or $\varsigma \leq \sigma$. This implies that ${\uparrow}\sigma \cap {\uparrow}\varsigma = {\uparrow}\varsigma$ or ${\uparrow}\sigma \cap {\uparrow}\varsigma = {\uparrow}\sigma$. Therefore, ${\uparrow}\sigma \cap {\uparrow}\varsigma$ is either $\varnothing$, ${\uparrow}\sigma$, or ${\uparrow}\varsigma$. This together with Lemma 3.4 yields that $\mathbf{T}_\kappa$ is coherent.

To see that $\mathbf{T}_\kappa$ is sober, let F be an irreducible closed set in $\mathbf{T}_\kappa$. Then F is a downset. We first show that F is a chain. If not, then there are $\sigma, \varsigma \in F$ that are unrelated. The set ${\downarrow}\sigma \cap {\downarrow}\varsigma$ has a $\leq$-greatest element (otherwise $\sigma|_n = \varsigma|_n$ for all $n \in \omega$, giving $\sigma = \varsigma$), say ς', which is finite. Therefore, the unique child of ς' that is under σ is not related to the unique child of ς' that is under ς. Thus, we may assume without loss of generality that both σ and ς are finite. But then $F \setminus {\uparrow}\sigma$ and $F \setminus {\uparrow}\varsigma$ are closed in $\mathbf{T}_\kappa$ and $F = (F \setminus {\uparrow}\sigma) \cup (F \setminus {\uparrow}\varsigma)$. This contradicts to F being irreducible, so F must be a chain.

We next show that $F = {\downarrow}\sigma$ for some $\sigma \in F$. If F contains an infinite sequence σ, then it is clear that $F = {\downarrow}\sigma$. Suppose that F contains no infinite sequences. Then since F is closed, F has a $\leq$-greatest element, say σ, yielding again that $F = {\downarrow}\sigma$. Because ${\downarrow}\sigma$ is the closure of σ in $\mathbf{T}_\kappa$, we conclude that F is the closure of a unique singleton. Thus, $\mathbf{T}_\kappa$ is sober, concluding the proof. $\square$

Remark 3.6. Recall that a poset $(P, \preceq)$ is a *directed complete partial order (DCPO)* provided that every directed set $D \subseteq P$ has a supremum. A *Scott open* subset U of a DCPO $(P, \preceq)$ is an upset such that for any directed set D, if $\sup D \in U$, then $D \cap U \neq \varnothing$. The set of Scott open subsets is a topology on P called the *Scott topology*. Clearly $\mathcal{T}_\kappa$ is a DCPO, and the Scott topology on T_κ is τ. Thus, $\mathbf{T}_\kappa$ can alternatively be thought of in terms of the Scott topology associated with $\mathcal{T}_\kappa$. This approach is utilized in [8, Sec. 6] for the binary tree with limits.

Definition 3.7.
1. Let $\mathfrak{T}_\kappa = (T_\kappa, \pi)$ where π is the patch topology of τ.
2. Let $\mathfrak{T}_\kappa^n$ be the subspace of $\mathfrak{T}_\kappa$ whose underlying set is T_κ^n.
3. Let $\mathfrak{T}_\kappa^\omega$ be the subspace of $\mathfrak{T}_\kappa$ whose underlying set is T_κ^ω.
4. Let $\mathfrak{T}_\kappa^\infty$ be the subspace of $\mathfrak{T}_\kappa$ whose underlying set is T_κ^∞.

Remark 3.8.
1. Since $\sigma \leq \varsigma$ iff σ belongs to the closure of $\{\varsigma\}$ in $\mathbf{T}_\kappa$, it follows that $(\mathfrak{T}_\kappa, \leq)$ is a Priestley space.
2. Viewing κ as a discrete space, we have:
   (a) The product space κ^ω is homeomorphic to T_κ^∞ as a subspace of $\mathbf{T}_\kappa$ since a basis for the product topology consists of sets of infinite sequences in which finitely many entries are fixed.
   (b) The product space κ^ω is also homeomorphic to $\mathfrak{T}_\kappa^\infty$ (see Theorem 3.13(1)).

We now exhibit a useful basis for $\mathfrak{T}_\kappa$. We do so in two lemmas.

Lemma 3.9. *A basis for $\mathfrak{T}_\kappa$ is given by sets of the form ${\uparrow}\sigma \setminus \bigcup_{i=0}^n {\uparrow}\varsigma_i$ where $\sigma, \varsigma_i \in T_\kappa^\omega$.*

Proof. Sets of the form $U \setminus V$ where U and V are compact open subsets of $\mathbf{T}_\kappa$ constitute a basis for the patch topology π. It follows from Lemma 3.4 that sets of the form

$$\left(\bigcup_{j=0}^m {\uparrow}\sigma_j\right) \setminus \left(\bigcup_{i=0}^n {\uparrow}\varsigma_i\right)$$

where $\sigma_j, \varsigma_i \in T_\kappa^\omega$ constitute a basis for π. Noting that

$$\left(\bigcup_{j=0}^m {\uparrow}\sigma_j\right) \setminus \left(\bigcup_{i=0}^n {\uparrow}\varsigma_i\right) = \bigcup_{j=0}^m \left({\uparrow}\sigma_j \setminus \bigcup_{i=0}^n {\uparrow}\varsigma_i\right)$$

completes the proof. □

Lemma 3.10. *The family $\left\{{\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda \mid \sigma \in T_\kappa^\omega \text{ and } \Lambda \subseteq \kappa \text{ is finite}\right\}$ is a basis for $\mathfrak{T}_\kappa$.*

Proof. By Lemma 3.9, it is sufficient to show that for each ${\uparrow}\varsigma \setminus \bigcup_{i=0}^n {\uparrow}\varsigma_i$, where $\varsigma, \varsigma_i \in T_\kappa^\omega$, and each $\sigma \in {\uparrow}\varsigma \setminus \bigcup_{i=0}^n {\uparrow}\varsigma_i$, there are $\rho \in T_\kappa^\omega$ and $\Lambda \subseteq \kappa$ finite such that $\sigma \in {\uparrow}\rho \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\rho.\lambda \subseteq {\uparrow}\varsigma \setminus \bigcup_{i=0}^n {\uparrow}\varsigma_i$. First suppose that σ is infinite. Let $m = \max\{\ell(\varsigma), \ell(\varsigma_i) \mid i = 0, \dots, n\}$.

Claim 3.11. ${\uparrow}(\sigma|_m) \subseteq {\uparrow}\varsigma \setminus \bigcup_{i=0}^n {\uparrow}\varsigma_i$.

Proof. Since $\sigma \in {\uparrow}\varsigma$ and $\ell(\varsigma) \le m$, we have that $\varsigma = \sigma|_{\ell(\varsigma)} \le \sigma|_m$. This implies that ${\uparrow}(\sigma|_m) \subseteq {\uparrow}\varsigma$. Suppose $\sigma' \in {\uparrow}(\sigma|_m) \cap {\uparrow}\varsigma_i$ for some $i \le n$. Then $\sigma|_m, \varsigma_i \in {\downarrow}\sigma'$, yielding that $\sigma|_m \le \varsigma_i$ or $\varsigma_i \le \sigma|_m$. Because $\ell(\varsigma_i) \le m$, it must be the case that $\varsigma_i \le \sigma|_m$. Since $\sigma|_m \le \sigma$, this gives that $\sigma \in {\uparrow}\varsigma_i$, a contradiction. Thus, ${\uparrow}(\sigma|_m) \cap {\uparrow}\varsigma_i = \varnothing$ for each $i \le n$, and hence ${\uparrow}(\sigma|_m) \subseteq {\uparrow}\varsigma \setminus \bigcup_{i=0}^n {\uparrow}\varsigma_i$. □

Taking $\rho = \sigma|_m$ and $\Lambda = \varnothing$ completes the proof for infinite σ. Next suppose that σ is finite. Let I be the subset of $\{0, \dots, n\}$ consisting of those i for which $\sigma \le \varsigma_i$. For each $i \in I$, there is a unique child $\sigma.\lambda_i$ of σ such that $\sigma.\lambda_i \le \varsigma_i$ (because $\sigma \ne \varsigma_i$). Put $\Lambda = \{\lambda_i \mid i \in I\}$. Then $\Lambda \subseteq \kappa$ is finite. Since each $\sigma.\lambda$ is a child of σ, we have that $\sigma \in {\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda$.

Claim 3.12. ${\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda \subseteq {\uparrow}\varsigma \setminus \bigcup_{i=0}^n {\uparrow}\varsigma_i$.

Proof. Let $\sigma' \in {\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda$. Then $\varsigma \le \sigma \le \sigma'$ and so $\sigma' \in {\uparrow}\varsigma$. If $\sigma' \in {\uparrow}\varsigma_i$ for some $i \le n$, then $\sigma, \varsigma_i \in {\downarrow}\sigma'$, so $\varsigma_i \le \sigma$ or $\sigma \le \varsigma_i$. Since $\sigma \notin \bigcup_{i=0}^n {\uparrow}\varsigma_i$, it must be the case that $\sigma \le \varsigma_i$. This yields that $i \in I$ and $\sigma \le \sigma.\lambda_i \le \varsigma_i \le \sigma'$. From this it follows that $\sigma' \in \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda$, a contradiction. Therefore, $\sigma' \notin \bigcup_{i=0}^n {\uparrow}\varsigma_i$, and hence ${\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda \subseteq {\uparrow}\varsigma \setminus \bigcup_{i=0}^n {\uparrow}\varsigma_i$. □

Since ${\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda$ is of the required form, the proof is complete. □

The next result presents some properties of an arbitrary $\mathfrak{T}_\kappa$, as well as some properties that depend on whether κ is finite or infinite.

Theorem 3.13.

(1) *For any $\sigma \in T_\kappa^\infty$, the family $\{{\uparrow}(\sigma|_n) \mid n \in \omega\}$ is a local basis of $\mathfrak{T}_\kappa$ at σ.*
(2) *If $\kappa \ge 2$, then $\mathfrak{T}_\kappa^\infty$ is crowded.*
(3) *The set T_κ^ω is dense in $\mathfrak{T}_\kappa$.*
(4) *If κ is finite, then*
  (a) *each $\sigma \in T_\kappa^\omega$ is isolated in $\mathfrak{T}_\kappa$;*
  (b) *$\mathfrak{T}_\kappa^\omega$ is discrete;*
  (c) *T_κ^∞ is closed in $\mathfrak{T}_\kappa$.*
(5) *If κ is infinite, then*
  (a) *T_κ^∞ is dense in $\mathfrak{T}_\kappa$;*

(b) *$\mathfrak{T}_\kappa$ is resolvable;*
(c) *$\mathfrak{T}_\kappa$ and $\mathfrak{T}_\kappa^\omega$ are crowded.*

(6) *If κ is countable, then $\mathfrak{T}_\kappa$ is metrizable.*
(7) *If κ is uncountable, then $\mathfrak{T}_\kappa$ is not metrizable.*

Proof. (1) This follows from Lemma 3.9 and Claim 3.11.

(2) This follows from (1) since ${\uparrow}(\sigma|_n)$ contains infinitely many infinite sequences for $\sigma \in T_\kappa^\infty$, $\kappa \geq 2$, and $n \in \omega$.

(3) This is clear from Lemma 3.10 since any basic set of the form ${\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda$, with $\sigma \in T_\kappa^\omega$ and $\Lambda \subseteq \kappa$ finite, contains σ.

(4a) For $\sigma \in T_\kappa^\omega$, we have $\{\sigma\} = {\uparrow}\sigma \setminus \bigcup_{\lambda\in\kappa} {\uparrow}\sigma.\lambda$. Thus, since κ is finite, $\{\sigma\}$ is open by Lemma 3.10.

(4b&c) These are immediate from (4a) since T_κ^ω consists of isolated points, giving that T_κ^ω is open, and hence $T_\kappa^\infty = T_\kappa \setminus T_\kappa^\omega$ is closed in $\mathfrak{T}_\kappa$.

(5a) This follows from Lemma 3.10 because for any basic open set ${\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda$, with $\sigma \in T_\kappa^\omega$ and $\Lambda \subseteq \kappa$ finite, we have $\kappa \setminus \Lambda \neq \varnothing$. Thus, such a basic open set has nonempty intersection with T_κ^∞.

(5b) By (3) and (5a), both T_κ^ω and T_κ^∞ are dense in $\mathfrak{T}_\kappa$. Since they are disjoint, it follows that $\mathfrak{T}_\kappa$ is resolvable.

(5c) Since $\mathfrak{T}_\kappa$ is resolvable, it is crowded. Therefore, so is $\mathfrak{T}_\kappa^\omega$ as it is a dense subspace of a crowded T_1-space.

(6) Suppose $(0 \neq)\kappa \leq \omega$. Then $\mathscr{S} = \{{\uparrow}\sigma \mid \sigma \in T_\kappa^\omega\}$ is countable, and so $\mathfrak{T}_\kappa$ is second-countable. This together with $\mathfrak{T}_\kappa$ being a Stone space implies that $\mathfrak{T}_\kappa$ is metrizable (see, e.g., [18, Thm. 4.2.8]).

(7) Suppose κ is uncountable. It is sufficient to show that $\mathfrak{T}_\kappa$ is not first-countable. Let $\mathscr{U} := \{U_n \mid n \in \omega\}$ be any (countable) collection of open neighborhoods of ε. It follows from Lemma 3.10 that for any $U_n \in \mathscr{U}$, all but finitely many children of ε are in U_n. Let C_n be the set of children of ε not contained in U_n. Then $C := \bigcup_{n\in\omega} C_n$ is countable. Because κ is uncountable, there is a child σ of ε such that $\sigma \notin C$. The set $U := {\uparrow}\varepsilon \setminus {\uparrow}\sigma$ is a basic open neighborhood of ε such that no $U_n \in \mathscr{U}$ satisfies $U_n \subseteq U$ (indeed, $\sigma \in U_n$ because $\sigma \notin C$, giving $U_n \not\subseteq U$). Thus, any local basis at ε must be uncountable, and hence $\mathfrak{T}_\kappa$ is not first-countable. □

We conclude this section by analyzing the case when κ is countable.

Theorem 3.14. *Let $(0 \neq)\kappa \leq \omega$.*

(1) *If $\kappa = 1$, then $\mathfrak{T}_\kappa$ is homeomorphic to the ordinal space $\omega + 1$.*
(2) *If $1 < \kappa < \omega$, then*
   (a) *$\mathfrak{T}_\kappa^\infty$ is homeomorphic to the Cantor space;*
   (b) *$\mathfrak{T}_\kappa$ is homeomorphic to the Pełczyński compactification of the discrete space ω.*
(3) *If $\kappa = \omega$, then*
   (a) *$\mathfrak{T}_\omega$ is an uncountable crowded completely metrizable Stone space of countable weight.*
   (b) *$\mathfrak{T}_\omega^\omega$ is homeomorphic to the space of rational numbers.*
   (c) *$\mathfrak{T}_\omega^\infty$ is homeomorphic to the Baire space.*
   (d) *$\mathfrak{T}_\omega^\infty$ is homeomorphic to the space of irrational numbers.*

Proof. (1) By Theorem 3.13(3 & 4b), $\mathfrak{T}_1^\omega$ is a dense discrete subspace of $\mathfrak{T}_1$. Since T_1^∞ is a singleton and T_1^ω is countably infinite, it follows that $\mathfrak{T}_1$ is homeomorphic to the one-point compactification of ω, which in turn is homeomorphic to $\omega + 1$.

(2a) By Theorem 3.13(4c), T_κ^∞ is closed in $\mathfrak{T}_\kappa$. As a closed subspace of the compact zero-dimensional metrizable space $\mathfrak{T}_\kappa$, we have that $\mathfrak{T}_\kappa^\infty$ is also compact, zero-dimensional, and metrizable. In addition, $\mathfrak{T}_\kappa^\infty$ is crowded by Theorem 3.13(2). Thus, by Brouwer's theorem (see, e.g., [18, Exercise 6.2.A(c)]), $\mathfrak{T}_\kappa^\infty$ is homeomorphic to the Cantor space.

(2b) By Theorem 3.13(3 & 4b), $\mathfrak{T}_\kappa^\omega$ is a countable dense discrete subspace of $\mathfrak{T}_\kappa$. Since the Pełczyński compactification is, up to homeomorphism, the compactification of the discrete space ω whose remainder is homeomorphic to the Cantor space, it follows that $\mathfrak{T}_\kappa$ is homeomorphic to the Pełczyński compactification.

(3a) Clearly $\mathfrak{T}_\omega$ is a Stone space (since the topology is the patch topology of $\mathbf{T}_\omega$), is uncountable (since $\omega \geq 2$), and has countable weight (since ω is countable). Moreover, $\mathfrak{T}_\omega$ is crowded by Theorem 3.13(5c). Because $\mathfrak{T}_\omega$ is metrizable, the result follows by recalling that a compact metrizable space is completely metrizable (see, e.g., [18, Thm. 4.3.28]).

(3b) Clearly T_ω^ω is countable, and $\mathfrak{T}_\omega^\omega$ is crowded and metrizable by Theorem 3.13(5c & 6). Thus, by Sierpiński's theorem (see, e.g., [18, Exercise 6.2.A(d)]), $\mathfrak{T}_\omega^\omega$ is homeomorphic to the space of rational numbers.

(3c) This follows from Remark 3.8(2b) since the Baire space is homeomorphic to the product of ω copies of the discrete space ω.

(3d) This follows from (3c) since the Baire space is homeomorphic to the space of irrational numbers (see, e.g., [18, Exercise 4.3.G]). $\square$

Remark 3.15. Despite the fact that $\mathfrak{T}_\omega^\omega$ is homeomorphic to the space of rational numbers and $\mathfrak{T}_\omega^\infty$ is homeomorphic to the space of irrational numbers, it is not the case that $\mathfrak{T}_\omega = \mathfrak{T}_\omega^\infty \cup \mathfrak{T}_\omega^\omega$ is homeomorphic to the space of real numbers. Indeed, $\mathfrak{T}_\omega$ is a Stone space, but the space of real numbers is not.

4. Mapping theorems for $\mathfrak{T}_\kappa$ and its subspaces for countable κ

In this section we construct a continuous map from $\mathfrak{T}_\kappa$ onto an arbitrary finite quasi-tree $\mathfrak{F}$ for large enough $\kappa \leq \omega$. It is built by combining a modification of the well-known unraveling technique with the labelling scheme introduced in [2, Sec. 4.1]. Given any finite rooted $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$, the scheme developed in [2] labels the infinite binary tree by elements of W so that $\mathfrak{F}$ is realized as an interior image of $\mathfrak{T}_2^\infty$. As the maps we construct are continuous and onto but not necessarily open, we subsequently explore when such a map is interior, as well as how to make it interior when it fails to be so.

4.1. **The basic construction.** Suppose $\mathfrak{F} = (W, R)$ is a finite quasi-tree and $|W|$ denotes the cardinality of W. We let κ be such that $\max(|W|, 2) \leq \kappa \leq \omega$. Choose and fix a root r of $\mathfrak{F}$. For each cluster $C \subseteq W$, choose and fix $w_C \in C$. For each $w \in W$, fix an enumeration $\{w_m \mid m < n_w\}$ of $R(w)$ such that $w_0 = w$.[2]

Recursive definition of $f : T_\kappa \to W$: Let $f(\varepsilon) = r$. Assuming that $f(\sigma) = w$ for some finite σ, let $f(\sigma.m) = w_{m \bmod n_w}$ in the chosen enumeration of $R(w) = R(f(\sigma))$ for $m < \kappa$. An inductive argument yields that $f(\sigma) R f(\varsigma)$ for finite sequences satisfying $\sigma \leq \varsigma$. Assume σ is infinite and $f(\sigma|_n) \in W$ for each $n \in \omega$. Then $\{f(\sigma|_n) \mid n \in \omega\}$ is an R-increasing sequence; that is, $f(\sigma|_n) R f(\sigma|_{n+1})$ for all $n \in \omega$. Because W is finite, there are $N \in \omega$ and a cluster $C \subseteq W$ such that $f(\sigma|_n) \in C$ whenever $n \geq N$. The sequence $\{f(\sigma|_n) \mid n \in \omega\}$ is either eventually constant in C or not. If the sequence $\{f(\sigma|_n) \mid n \in \omega\}$ is eventually constant with value w, let $f(\sigma) = w$. Otherwise, let $f(\sigma) = w_C$. It is worth pointing out for infinite σ that we have $f(\sigma|_n) R f(\sigma)$ for all $n \in \omega$. A straightforward transfinite induction on the length of sequences yields that f is well defined.

[2]While this requirement is unnecessary, it makes Claims 4.6, 4.8, and 4.10 simpler.

Lemma 4.1. *Let $w \in W$. Then $f^{-1}(R(w)) = \bigcup_{\sigma\in\Sigma} {\uparrow}\sigma$, where $\Sigma = \{\sigma \in T_\kappa^\omega \mid wRf(\sigma)\}$.*

Proof. Let $\varsigma \in \bigcup_{\sigma\in\Sigma} {\uparrow}\sigma$. Then there is $\sigma \in T_\kappa^\omega$ such that $wRf(\sigma)$ and $\varsigma \in {\uparrow}\sigma$. Since $\sigma \le \varsigma$, we have that $wRf(\sigma)Rf(\varsigma)$. Because R is transitive, $f(\varsigma) \in R(w)$, yielding $\varsigma \in f^{-1}(R(w))$.

Conversely, let $\varsigma \in f^{-1}(R(w))$, so $wRf(\varsigma)$. If ς is finite, then $\varsigma \in \Sigma$ and $\varsigma \in {\uparrow}\varsigma \subseteq \bigcup_{\sigma\in\Sigma} {\uparrow}\sigma$. Assume ς is infinite. By the definition of f, there is $N \in \omega$ such that $f(\varsigma|_n)$ is in the cluster of $f(\varsigma)$ for all $n \ge N$. Therefore, $wRf(\varsigma)Rf(\varsigma|_N)$. Since R is transitive, we have that $wRf(\varsigma|_N)$, and so $\varsigma|_N \in \Sigma$. As $\varsigma|_N \le \varsigma$, it follows that $\varsigma \in {\uparrow}(\varsigma|_N) \subseteq \bigcup_{\sigma\in\Sigma} {\uparrow}\sigma$. □

Theorem 4.2. *The function $f : T_\kappa \to W$ is a continuous mapping of $\mathfrak{T}_\kappa$ onto $\mathfrak{F}$.*

Proof. By Lemmas 3.10 and 4.1, $f^{-1}(R(w))$ is open in $\mathfrak{T}_\kappa$ for every $w \in W$. Thus, f is continuous. To see that f is onto, since $|W| \le \kappa$, we have that $f(\{\varepsilon.m \mid m < |W|\} = R(r) = W$. □

Remark 4.3. If κ is finite, then f need not be open. To see this, let $\kappa \ge 2$ be finite and let $\mathfrak{F} = (W, R)$ be a finite quasi-tree with $|W| \ge 2$. By Theorem 3.13(4a), ε is an isolated point of $\mathfrak{T}_\kappa$, but $f(\{\varepsilon\}) = \{r\}$ is not open in $\mathfrak{F}$. Thus, f is not an open mapping.

4.2. **The case $\kappa = \omega$.** We next show that if $\kappa = \omega$, then f is an interior surjection, and so are its restrictions to $\mathfrak{T}_\omega^\omega$ and $\mathfrak{T}_\omega^\infty$.

Theorem 4.4.

(1) *The function $f : T_\omega \to W$ is an interior mapping of $\mathfrak{T}_\omega$ onto $\mathfrak{F}$.*
(2) *The restriction $g := f|_{T_\omega^\omega}$ is an interior mapping of $\mathfrak{T}_\omega^\omega$ onto $\mathfrak{F}$.*
(3) *The restriction $h := f|_{T_\omega^\infty}$ is an interior mapping of $\mathfrak{T}_\omega^\infty$ onto $\mathfrak{F}$.*

Proof. (1) By Theorem 4.2, we only need to see that f is open. For this, by Lemma 3.10, it is sufficient to show that

$$f\left({\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda\right) = R(f(\sigma))$$

for arbitrary $\sigma \in T_\omega^\omega$ and finite $\Lambda \subseteq \omega$. Let $w \in f\left({\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda\right)$. There is $\varsigma \in {\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda$ such that $f(\varsigma) = w$. Because $\sigma \le \varsigma$, we have that $f(\sigma)Rf(\varsigma)$. Therefore, $w = f(\varsigma) \in R(f(\sigma))$. Conversely, let $w \in R(f(\sigma))$. In the enumeration of $R(f(\sigma))$, we have that $w = w_m$ for some $m < n_{f(\sigma)}$. Since Λ is finite, there is $M \in \omega$ such that $M \notin \Lambda$ and $M \bmod n_{f(\sigma)} = m$. We have that $\sigma.M \in {\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda$ and $f(\sigma.M) = w_{M \bmod n_{f(\sigma)}} = w_m = w$. Thus, $w \in f\left({\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda\right)$, and so f is open.

(2) Since g is a restriction of f and f is continuous, so is g. To see that g is open and onto, we use the following claim.

Claim 4.5. *For arbitrary $\sigma \in T_\omega^\omega$ and finite $\Lambda \subseteq \omega$,*

$$g\left(\left({\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda\right) \cap T_\omega^\omega\right) = R(g(\sigma)).$$

Proof. Since g is the restriction of f to T_ω^ω, we clearly have

$$g\left(\left({\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda\right) \cap T_\omega^\omega\right) \subseteq f\left({\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda\right) = R(f(\sigma)) = R(g(\sigma)).$$

The proof of the $\supseteq$ direction is the same as for f. □

It follows from Lemma 3.10 and Claim 4.5 that g is open. Furthermore, Claim 4.5 gives that g is onto since

$$g(T_\omega^\omega) = g({\uparrow}\varepsilon \cap T_\omega^\omega) = g\left(\left({\uparrow}\varepsilon \setminus \bigcup_{\lambda\in\varnothing} {\uparrow}\varepsilon.\lambda\right) \cap T_\omega^\omega\right) = R(g(\varepsilon)) = R(r) = W.$$

(3) Since h is a restriction of f and f is continuous, so is h. To see that h is open and onto, we modify the proof of (2).

Claim 4.6. *For arbitrary $\sigma \in T_\omega^\omega$ and finite $\Lambda \subseteq \omega$,*
$$h\left(\left(\uparrow\sigma \setminus \bigcup\nolimits_{\lambda\in\Lambda} \uparrow\sigma.\lambda\right) \cap T_\omega^\infty\right) = R(f(\sigma)).$$

Proof. Obviously we have that
$$h\left(\left(\uparrow\sigma \setminus \bigcup\nolimits_{\lambda\in\Lambda} \uparrow\sigma.\lambda\right) \cap T_\omega^\infty\right) \subseteq f\left(\uparrow\sigma \setminus \bigcup\nolimits_{\lambda\in\Lambda} \uparrow\sigma.\lambda\right) = R(f(\sigma)).$$
Let $w \in R(f(\sigma))$ be w_m for some $m < n_{f(\sigma)}$. There is $M \in \omega\setminus\Lambda$ such that $M \bmod n_{f(\sigma)} = m$, $\sigma.M \in \uparrow\sigma\setminus\bigcup_{\lambda\in\Lambda}\uparrow\sigma.\lambda$, and $f(\sigma.M) = w$. Recalling that $w = w_0$ in the enumeration of $R(w)$, we define $\varsigma : \omega \to \omega$ by
$$\varsigma(n) = \begin{cases} \sigma(n) & n < \ell(\sigma) \\ M & n = \ell(\sigma) \\ 0 & n > \ell(\sigma) \end{cases}$$
Then $\varsigma \in T_\omega^\infty$, $\varsigma|_{\ell(\sigma)} = \sigma$, and $\varsigma|_{\ell(\sigma)+1} = \sigma.M$, which yields that $\varsigma \in (\uparrow\sigma \setminus \bigcup_{\lambda\in\Lambda}\uparrow\sigma.\lambda) \cap T_\omega^\infty$. Since $w = w_0$ in the enumeration of $R(w)$, it follows from the definition of f that $f(\varsigma|_{\ell(\sigma)}) = f(\sigma)$ and $f(\varsigma|_n) = w$ for all $n \geq \ell(\sigma) + 1$. Thus, $w = f(\varsigma) = h(\varsigma) \in h\left(\left(\uparrow\sigma \setminus \bigcup_{\lambda\in\Lambda}\uparrow\sigma.\lambda\right) \cap T_\omega^\infty\right)$. □

That h is open follows from Lemma 3.10 and Claim 4.6. Moreover, Claim 4.6 also implies that
$$h(T_\omega^\infty) = h(\uparrow\varepsilon \cap T_\omega^\infty) = h\left(\left(\uparrow\varepsilon \setminus \bigcup\nolimits_{\lambda\in\varnothing} \uparrow\varepsilon.\lambda\right) \cap T_\omega^\infty\right) = R(f(\varepsilon)) = R(r) = W,$$
yielding that h is onto. □

4.3. **The case $\kappa < \omega$.** It is obvious that if $|W| = 1$, then each of f, $f|_{T_\kappa^\omega}$, and $f|_{T_\kappa^\infty}$ is an interior surjection. Suppose $|W| \geq 2$. Then, as we saw in Remark 4.3, f is not open, and neither is $f|_{T_\kappa^\omega}$. On the other hand, we show that $f|_{T_\kappa^\infty}$ is an interior surjection. The proof is similar to that of Theorem 4.4(3).

Theorem 4.7. *Let $\kappa < \omega$. The restriction $h := f|_{T_\kappa^\infty}$ is an interior mapping of $\mathfrak{T}_\kappa^\infty$ onto $\mathfrak{F}$.*

Proof. The following claim, which is analogous to [2, Lem. 4.4], is the crux of the proof.

Claim 4.8. *For $\sigma \in T_\kappa^\omega$, we have that*
$$h(\uparrow\sigma \cap T_\kappa^\infty) = R(f(\sigma)).$$

Proof. We have that $h(\uparrow\sigma \cap T_\kappa^\infty) \subseteq f(\uparrow\sigma) \subseteq R(f(\sigma))$ since $f(\sigma)Rf(\varsigma)$ whenever $\sigma \leq \varsigma$. Let $w \in R(f(\sigma))$. Then $w = w_m$ in the enumeration of $R(f(\sigma))$ for some $m < n_{f(\sigma)} \leq |W| \leq \kappa$. Define $\varsigma : \omega \to \kappa$ by
$$\varsigma(n) = \begin{cases} \sigma(n) & n < \ell(\sigma) \\ m & n = \ell(\sigma) \\ 0 & n > \ell(\sigma) \end{cases}$$
Then $\varsigma \in T_\kappa^\infty$, $\sigma \leq \varsigma$, and $f(\varsigma|_n) = w$ for all $n \geq \ell(\sigma)$. Thus, $h(\varsigma) = f(\varsigma) = w$, giving that $w \in h(\uparrow\sigma \cap T_\kappa^\infty)$. □

For $\sigma \in T_\kappa^\omega$ and $\Lambda \subseteq \kappa$, we have:
$$\begin{aligned} \left(\uparrow\sigma \setminus \bigcup\nolimits_{\lambda\in\Lambda} \uparrow\sigma.\lambda\right) \cap T_\kappa^\infty &= \left(\{\sigma\} \cup \bigcup\nolimits_{\lambda\in\kappa\setminus\Lambda} \uparrow\sigma.\lambda\right) \cap T_\kappa^\infty \\ &= \underbrace{(\{\sigma\} \cap T_\kappa^\infty)}_{\varnothing} \cup \bigcup\nolimits_{\lambda\in\kappa\setminus\Lambda} (\uparrow\sigma.\lambda \cap T_\kappa^\infty) \end{aligned}$$

By Claim 4.8,

$$\begin{aligned} h\left(\left(\uparrow\sigma \setminus \bigcup_{\lambda\in\Lambda} \uparrow\sigma.\lambda\right) \cap T_\kappa^\infty\right) &= h\left(\bigcup_{\lambda\in\kappa\setminus\Lambda} (\uparrow\sigma.\lambda \cap T_\kappa^\infty)\right) \\ &= \bigcup_{\lambda\in\kappa\setminus\Lambda} h\left(\uparrow\sigma.\lambda \cap T_\kappa^\infty\right) = \bigcup_{\lambda\in\kappa\setminus\Lambda} R(f(\sigma.\lambda)). \end{aligned}$$

Thus, $h\left(\left(\uparrow\sigma \setminus \bigcup_{\lambda\in\Lambda} \uparrow\sigma.\lambda\right) \cap T_\kappa^\infty\right)$ is open in $\mathfrak{F}$, yielding that h is interior. Finally, to see that h is onto, observe that

$$h(T_\kappa^\infty) = h(\uparrow\varepsilon \cap T_\kappa^\infty) = R(f(\varepsilon)) = R(r) = W.$$

□

4.4. **Interior mappings onto finite top-thin-quasi-trees.** Although for finite $\kappa \geq 2$, the continuous surjection $f : \mathfrak{T}_\kappa \to \mathfrak{F}$ is not open (see Remark 4.3), we next show that if $\mathfrak{F}$ is a finite top-thin-quasi-tree, then there is a finite $\kappa \geq 2$ such that $\mathfrak{F}$ is an interior image of $\mathfrak{T}_\kappa$.

Theorem 4.9. *Let $\mathfrak{F} = (W, R)$ be a finite top-thin-quasi-tree obtained from $\mathfrak{G} = (V, Q)$. There is $2 \leq \kappa < \omega$ such that $\mathfrak{F}$ is an interior image of $\mathfrak{T}_\kappa$.*

Proof. If V is a singleton, then $\mathfrak{F}$ is isomorphic to the chain consisting of two points. Take $\kappa = 2$ and define $f : T_2 \to W$ by $f(T_2^\infty) = V$ and $f(T_2^\omega) = W \setminus V$. Since T_2^∞ and T_2^ω are complements and W consists of two points, f is a well-defined surjection. We have that f is continuous since $f^{-1}(W \setminus V) = T_2^\omega = \mathsf{Iso}(\mathfrak{T}_2)$ is open in $\mathfrak{T}_2$ (see Theorem 3.13(4a)), and $W \setminus V$ is the only proper nonempty open set in $\mathfrak{F}$. Let U be a nonempty open subset of $\mathfrak{T}_2$. If $U \subseteq T_2^\omega$, then $f(U) = W \setminus V$ is open in $\mathfrak{F}$. If $U \not\subseteq T_2^\omega$, then $f(U) = W$ since T_2^ω is dense in $\mathfrak{T}_2$ (see Theorem 3.13(3)). Thus, f is open and hence interior.

Suppose V consists of $\kappa \geq 2$ points. We may apply the basic construction to the finite quasi-tree $\mathfrak{G}$, yielding a continuous surjection $g : \mathfrak{T}_\kappa \to \mathfrak{G}$ such that $g(\varepsilon) = r$ is a root of $\mathfrak{G}$, and hence a root of $\mathfrak{F}$. Note that g is neither open (by Remark 4.3) nor onto (since $g(T_\kappa) = V \neq W$). The idea is to define $f : T_\kappa \to W$ by changing the values of g on finite sequences, i.e. the isolated points of $\mathfrak{T}_\kappa$. For each $v \in V$, choose $w_v \in W \setminus V = \max(\mathfrak{F})$ such that vRw_v. Define $f : T_\kappa \to W$ by setting

$$f(\sigma) = \begin{cases} g(\sigma) & \text{if } \sigma \in T_\kappa^\infty \\ w_{g(\sigma)} & \text{if } \sigma \in T_\kappa^\omega \end{cases}$$

Claim 4.10. *For $\sigma \in T_\kappa^\omega$, we have $f(\uparrow\sigma) = R(g(\sigma))$.*

Proof. Let $\varsigma \in \uparrow\sigma$. Then $g(\sigma)Rg(\varsigma)$ since Q is the restriction of R to V. If $\varsigma \in T_\kappa^\infty$, then $g(\sigma)Rg(\varsigma) = f(\varsigma)$. If $\varsigma \in T_\kappa^\omega$, then $g(\sigma)Rg(\varsigma)Rw_{g(\varsigma)} = f(\varsigma)$. In either case we have that $f(\varsigma) \in R(g(\sigma))$, hence $f(\uparrow\sigma) \subseteq R(g(\sigma))$. Conversely, let $w \in R(g(\sigma))$. To show that $w \in f(\uparrow\sigma)$, there are two cases to consider: either $w \in W \setminus V$ or $w \in V$.

First suppose that $w \in W \setminus V$. Because $\mathfrak{F}$ is a top-thin-quasi-tree, there is $v \in \mathrm{qmax}(\mathfrak{G})$ such that vRw. Since $\mathfrak{F}$ is a quasi-tree in which $v, g(\sigma) \in R^{-1}(w)$, either $vRg(\sigma)$ or $g(\sigma)Rv$, giving that $vQg(\sigma)$ or $g(\sigma)Qv$. Recalling that v is quasi-maximal in $\mathfrak{G}$, we have $v \in Q(g(\sigma))$ in both cases. By the definition of g, there is $\lambda < \kappa$ such that $g(\sigma.\lambda) = v$. Since $v \in \mathrm{qmax}(\mathfrak{G})$ and $\mathfrak{F}$ is a top-thin-quasi-tree obtained from $\mathfrak{G}$, it must be the case that $w_v = w$. Therefore, $f(\sigma.\lambda) = w_{g(\sigma.\lambda)} = w_v = w$. As $\sigma.\lambda \in \uparrow\sigma$, we conclude that $w \in f(\uparrow\sigma)$.

Next suppose that $w \in V$. Then $w \in Q(g(\sigma))$. So, by the definition of g, there is $m < \kappa$ such that $g(\sigma.m) = w$; and for any $\sigma' \in T_\kappa^\omega$ with $g(\sigma') = w$, we have that $g(\sigma'.0) = w_0 = w$.

Define $\varsigma : \omega \to \kappa$ by

$$\varsigma(n) = \begin{cases} \sigma(n) & \text{if } n < \ell(\sigma) \\ m & \text{if } n = \ell(\sigma) \\ 0 & \text{if } n > \ell(\sigma) \end{cases}$$

Then $\varsigma \in {\uparrow}\sigma$ and $g(\varsigma|_n) = w$ for all $n \geq \ell(\sigma)$. Therefore, $f(\varsigma) = g(\varsigma) = w$. Thus, $w \in f({\uparrow}\sigma)$. ☐

Claim 4.10 immediately yields that f is onto since $f({\uparrow}\varepsilon) = R(g(\varepsilon)) = R(r) = W$. Utilizing Lemma 3.10 and Claim 4.10, we show that f is open. Let $\sigma \in T_\kappa^\omega$ and $\Lambda \subseteq \kappa$. Then

$$\begin{aligned} f\left({\uparrow}\sigma \setminus \bigcup\nolimits_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda\right) &= f\left(\{\sigma\} \cup \bigcup\nolimits_{\lambda\in\kappa\setminus\Lambda} {\uparrow}\sigma.\lambda\right) \\ &= \{f(\sigma)\} \cup \bigcup\nolimits_{\lambda\in\kappa\setminus\Lambda} f\left({\uparrow}\sigma.\lambda\right) = \{f(\sigma)\} \cup \bigcup\nolimits_{\lambda\in\kappa\setminus\Lambda} R(g(\sigma.\lambda)) \end{aligned}$$

is open in $\mathfrak{F}$ since $f(\sigma) \in \max(\mathfrak{F})$.

To see that f is continuous, let $w \in W$ and $\sigma \in f^{-1}(R(w))$. If σ is finite, then σ is an isolated point, hence an interior point of $f^{-1}(R(w))$. Suppose σ is infinite. By definition of g, there is $N \in \omega$ such that for all $n \geq N$, $g(\sigma|_n)$ is in the same cluster as $g(\sigma)$. Since $f(\sigma) \in R(w)$ and $f(\sigma) = g(\sigma)$, Claim 4.10 gives that

$$f\left({\uparrow}\sigma|_N\right) = R(g(\sigma|_N)) = R(g(\sigma)) = R(f(\sigma)) \subseteq R(w).$$

We conclude that σ is an interior point of $f^{-1}(R(w))$ since ${\uparrow}\sigma|_N$ is open, contains σ, and is contained in $f^{-1}(R(w))$. Therefore, $f^{-1}(R(w))$ is open in $\mathfrak{T}_\kappa$, and so f is continuous. ☐

4.5. **Trees of finite depth.** If we consider the subspace $\mathfrak{T}_\kappa^n$ of $\mathfrak{T}_\kappa$, then our basic construction always fails to deliver an open mapping except in the trivial case when $\mathfrak{F} = (W, R)$ consists of a single point. Indeed, for any $n \in \omega$, there is a sequence σ of length n such that $f(\sigma) = r$. By Lemma 3.10, such σ is an isolated point of $\mathfrak{T}_\kappa^n$ because $\{\sigma\} = {\uparrow}\sigma \cap T_\kappa^n$. Therefore, $f(\{\sigma\}) = \{r\}$ is not open in $\mathfrak{F}$, showing that f is not open.

If κ is finite, then the space $\mathfrak{T}_\kappa^n$ is discrete (since $\mathfrak{T}_\kappa^\omega$ is discrete by Theorem 3.13(4b)). So for any $n \in \omega$, the one-point quasi-tree is the only interior image of $\mathfrak{T}_\kappa^n$. In case $\kappa = \omega$, we show that the tree $\mathcal{T}_\omega^n$ is an interior image of $\mathfrak{T}_\omega^n$. For this we utilize the following lemma, whose straightforward proof we leave out.

Lemma 4.11. *Let X, Y be topological spaces and $f : X \to Y$ an onto interior map. Suppose $C \subseteq Y$ and $D = f^{-1}(C)$. Then the restriction of f to D is an interior mapping onto C.*

Theorem 4.12.
1. *The Alexandroff space $\mathcal{T}_\omega^\omega$ is an interior image of $\mathfrak{T}_\omega^\omega$.*
2. *For $n \in \omega$, the Alexandroff space $\mathcal{T}_\omega^n$ is an interior image of $\mathfrak{T}_\omega^n$.*

Proof. (1) Let $\{K_n \mid n \in \omega\}$ be a partition of ω such that each K_n is infinite. Recursively define $f : T_\omega^\omega \to T_\omega^\omega$ by $f(\varepsilon) = \varepsilon$ and $f(\sigma.m) = f(\sigma).n$ whenever $f(\sigma)$ is defined and $m \in K_n$. A straightforward inductive argument on the length of $\sigma \in T_\omega^\omega$ shows that f is a well-defined onto mapping such that for each $\sigma \in T_\omega^\omega$, $f(\{\sigma.m \mid m \in \omega\}) = \{f(\sigma).n \mid n \in \omega\}$ and $\ell(\sigma) = \ell(f(\sigma))$. This yields that f is a p-morphism, and hence an interior mapping of $\mathcal{T}_\omega^\omega$ onto $\mathcal{T}_\omega^\omega$. Since the Alexandroff topology of $\mathcal{T}_\omega^\omega$ is coarser than the topology of $\mathfrak{T}_\omega^\omega$, it follows that f is a continuous mapping from $\mathfrak{T}_\omega^\omega$ onto $\mathcal{T}_\omega^\omega$.

We show that f is open. By Lemma 3.10, sets of the form

$$\left({\uparrow}\sigma \setminus \bigcup\nolimits_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda\right) \cap T_\omega^\omega$$

constitute a basis for $\mathfrak{T}_\omega^\omega$ where $\sigma \in T_\omega^\omega$ and $\Lambda \subseteq \omega$ is finite. Consider $U = {\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda$. Since $K_n \setminus \Lambda \neq \varnothing$ for each $n \in \omega$, we have:

$$f\left(\{\sigma.m \mid m \in \omega \setminus \Lambda\}\right) = f\left(\left\{\sigma.m \mid m \in \bigcup\nolimits_{n\in\omega} K_n \setminus \Lambda\right\}\right) = \{f(\sigma).n \mid n \in \omega\}.$$

This yields that $f(U \cap T_\omega^\omega) = {\uparrow}f(\sigma) \cap T_\omega^\omega$. Thus, f is open.

(2) Let $n \in \omega$. Since $\ell(\sigma) = \ell(f(\sigma))$ for each $\sigma \in T_\omega^\omega$ and f is onto, it follows that $f^{-1}(T_\omega^n) = T_\omega^n$. Thus, the restriction of f to $T_\omega^n = f^{-1}(T_\omega^n)$ is an interior mapping of $\mathfrak{T}_\omega^n$ onto $\mathcal{T}_\omega^n$ by Lemma 4.11. □

5. Topological completeness via trees for S4, S4.1, Grz, and Grz_n

In this section we give alternative proofs of some well-known topological completeness results utilizing the mapping theorems of the previous section.

5.1. **Completeness for S4.** We first focus on the spaces $\mathfrak{T}_\omega$, $\mathfrak{T}_\omega^\omega$, and $\mathfrak{T}_\kappa^\infty$ for $2 \le \kappa \le \omega$. Since each of these spaces is crowded and metrizable, it follows from the McKinsey-Tarski theorem that the logic of any of these spaces is S4. We give an alternate proof of these results by utilizing mapping theorems of Section 4 in conjunction with the fact that S4 is the logic of finite quasi-trees. As a result, we obtain new proofs that S4 is the logic of the Cantor space, the space of rational numbers, the space of irrational numbers, and the Baire space. We refer to [23, 2] for alternate proofs that S4 is the logic of the Cantor space, and to [3] for an alternate proof that S4 is the logic of the space of rational numbers.

Theorem 5.1.
(1) *For $X \in \{\mathfrak{T}_\omega, \mathfrak{T}_\omega^\omega, \mathfrak{T}_\kappa^\infty \mid 2 \le \kappa \le \omega\}$, the logic of X is* S4.
(2) S4 *is the logic of the Cantor space.*
(3) S4 *is the logic of the space of rational numbers.*
(4) S4 *is the logic of the Baire space.*
(5) S4 *is the logic of the space of irrational numbers.*

Proof. (1) Suppose that $\mathsf{S4} \nvdash \varphi$. Then φ is refuted on some finite quasi-tree $\mathfrak{F}$. By Theorems 4.4 and 4.7, $\mathfrak{F}$ is an interior image of X. Since interior images reflect refutations, X refutes φ. Therefore, φ is not a theorem of the logic of X, proving the result.

(2) This follows from (1) and Theorem 3.14(2a).
(3) This follows from (1) and Theorem 3.14(3b).
(4) This follows from (1) and Theorem 3.14(3c).
(5) This follows from (1) and Theorem 3.14(3d). □

5.2. **Completeness for S4.1.** We next give a new proof of [11, Cor. 3.19] that S4.1 is the logic of the Pełczyński compactification of the discrete space ω. Again we use a mapping theorem of Section 4 and that S4.1 is the logic of finite top-thin-quasi-trees.

Theorem 5.2.
(1) S4.1 *is the logic of the Pełczyński compactification of ω.*
(2) S4.1 *is the logic of $\mathfrak{T}_\kappa$ for finite $\kappa \ge 2$.*

Proof. (1) Let X be the Pełczyński compactification of ω. Since X is densely discrete, $\mathsf{S4.1} \subseteq \mathsf{Log}(X)$. Suppose $\mathsf{S4.1} \nvdash \varphi$. Then there is a finite top-thin-quasi-tree $\mathfrak{F}$ that refutes φ. By Theorem 4.9, there is $2 \le \kappa < \omega$ such that $\mathfrak{F}$ is an interior image of $\mathfrak{T}_\kappa$. By Theorem 3.14(2b), $\mathfrak{T}_\kappa$ and X are homeomorphic. Therefore, $\mathfrak{F}$ is an interior image of X. As interior images reflect refutations, $X \nvDash \varphi$. Thus, $\mathsf{S4.1} \supseteq \mathsf{Log}(X)$, completing the proof.

(2) This follows from (1) and Theorem 3.14(2b). □

5.3. **Completeness for Grz and Grz_n.** Finally, we give new proofs of well-known topological completeness results for Grz and Grz_{n+1} for $n \in \omega$. For this we require several lemmas.

Lemma 5.3. *For each $n \in \omega$, the space $\mathfrak{T}_\omega^n$ is a Stone space.*

Proof. Since $\mathfrak{T}_\omega$ is a Stone space (see Theorem 3.14(3a)), it is sufficient to show that T_ω^n is closed in $\mathfrak{T}_\omega$. Let $A = \{\sigma \in T_\omega \mid \ell(\sigma) = n+1\}$. Then $T_\omega \setminus T_\omega^n = \bigcup_{\sigma \in A} {\uparrow}\sigma$ is open in $\mathfrak{T}_\omega$. Thus, T_ω^n is closed in $\mathfrak{T}_\omega$. □

Lemma 5.4. *Let $n \in \omega$.*

(1) $\mathsf{Iso}(\mathfrak{T}_\omega^n) = \{\sigma \in T_\omega^n \mid \ell(\sigma) = n\}$.
(2) $\mathsf{d}(T_\omega^{n+1}) = T_\omega^n$.

Proof. (1) Let $\sigma \in T_\omega^n$. If $\ell(\sigma) = n$, then $\{\sigma\} = {\uparrow}\sigma \cap T_\omega^n$ is open in $\mathfrak{T}_\omega^n$ since ${\uparrow}\sigma$ is a basic open neighborhood of σ in $\mathfrak{T}_\omega$. Therefore, σ is an isolated point of $\mathfrak{T}_\omega^n$. Suppose that $\ell(\sigma) < n$ and U is an open neighborhood of σ in $\mathfrak{T}_\omega$. By Lemma 3.10, all but finitely many children of σ are in U. Since the length of a child of σ is $\ell(\sigma) + 1 \leq n$, $U \cap T_\omega^n$ is not a singleton. Thus, σ is not an isolated point of $\mathfrak{T}_\omega^n$.

(2) By (1), we have:

$$\mathsf{d}(T_\omega^{n+1}) = T_\omega^{n+1} \setminus \mathsf{Iso}(T_\omega^{n+1}) = T_\omega^{n+1} \setminus \{\sigma \in T_\omega^{n+1} \mid \ell(\sigma) = n+1\} = T_\omega^n.$$

□

Let X be a compact scattered space. Since X is scattered, there is a least ordinal β such that $\mathsf{d}^\beta(X) = \varnothing$. Because X is compact, we have $\beta = \alpha + 1$ and $\mathsf{d}^\alpha(X)$ is finite. Call (α, m) the *characteristic system* of X where m is the cardinality of $\mathsf{d}^\alpha(X)$; see [27, Def. 8.6.8].

Theorem 5.5. *For $n \in \omega$, the space $\mathfrak{T}_\omega^n$ is homeomorphic to the ordinal space $\omega^n + 1$.*

Proof. It follows by an inductive argument based on Lemma 5.4(2) that $\mathsf{d}^n(T_\omega^n) = \{\varepsilon\}$ and $\mathsf{d}^{n+1}(T_\omega^n) = \varnothing$. Therefore, $\mathfrak{T}_\omega^n$ is a compact Hausdorff scattered space whose characteristic system is $(n, 1)$. It follows from Lemma 3.10 that $\mathfrak{T}_\omega^n$ is second countable, and hence first countable. Therefore, $\mathfrak{T}_\omega^n$ is homeomorphic to an ordinal space by the Mazurkiewicz-Sierpiński theorem (see, e.g., [27, Thm. 8.6.10]). Thus, since the characteristic system of $\mathfrak{T}_\omega^n$ is $(n, 1)$, $\mathfrak{T}_\omega^n$ is homeomorphic to $\omega^n + 1$ by [27, Prop. 8.6.9]. □

Remark 5.6. An immediate consequence of Theorem 5.5 is that $\mathfrak{T}_\omega^1$ is homeomorphic to the one-point compactification of the discrete space ω. For $n \in \omega$, the space $\omega^{n+1} + 1$ can obtained as an adjunction space from $\omega^n + 1$ and countably many copies of $\omega + 1$ by gluing each isolated point of $\omega^n + 1$ to the limit point of a copy of $\omega + 1$. Analogously, $\mathfrak{T}_\omega^{n+1}$ can be obtained as an adjunction space from $\mathfrak{T}_\omega^n$ and countably many copies of $\mathfrak{T}_\omega^1$ by gluing each isolated point (leaf) of $\mathfrak{T}_\omega^n$ to the limit point (root) of a copy of $\mathfrak{T}_\omega^1$. For $n = 1$, these adjunctions are depicted in Figure 3.

We are ready to give an alternate proof that Grz_{n+1} is the logic of the ordinal space $\omega^n + 1$ [1] (see also [12]). For this we recall that Grz_{n+1} is the logic of finite trees of depth $\leq n+1$ (see Table 3). Since each such tree is a p-morphic image of $\mathcal{T}_\omega^n$, which is of depth $n+1$, we have that Grz_{n+1} is the logic of $\mathcal{T}_\omega^n$.

Theorem 5.7. *Let $n \in \omega$.*

(1) *The logic of $\mathfrak{T}_\omega^n$ is Grz_{n+1}.*
(2) *The logic of the ordinal space $\omega^n + 1$ is Grz_{n+1}.*

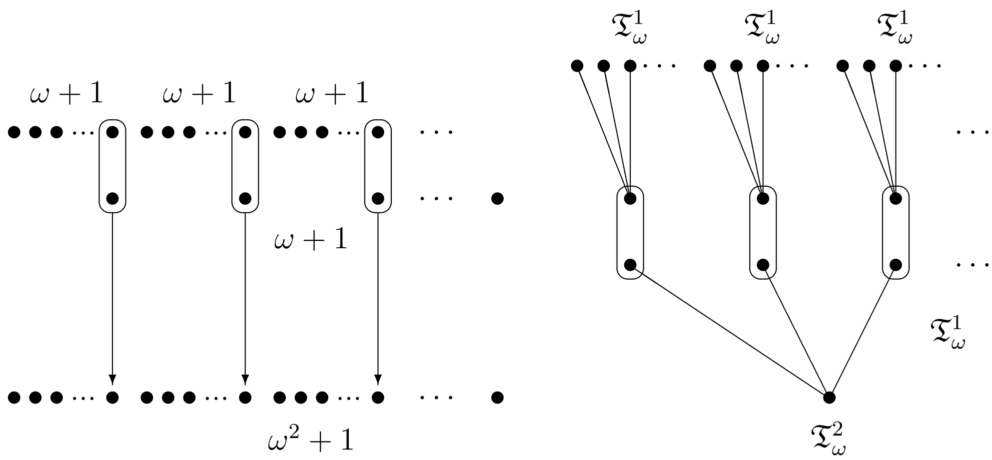

FIGURE 3. Realizing ω^2+1 and $\mathfrak{T}^2_\omega$ as adjunction spaces.

Proof. (1) Since $\mathfrak{T}^n_\omega$ is a scattered space such that $r(\mathfrak{T}^n_\omega) = n+1$, it follows from [7, Lem. 3.6] that Grz_{n+1} is contained in the logic of $\mathfrak{T}^n_\omega$. Conversely, if $\mathsf{Grz}_{n+1} \nvdash \varphi$, then φ is refuted on $\mathcal{T}^n_\omega$. Since Theorem 4.12(2) gives that $\mathcal{T}^n_\omega$ is an interior image of $\mathfrak{T}^n_\omega$ and interior images reflect refutations, $\mathfrak{T}^n_\omega$ refutes φ. Thus, the logic of $\mathfrak{T}^n_\omega$ is contained in Grz_{n+1}, and the equality follows.

(2) This follows from (1) and Theorem 5.5. □

Corollary 5.8. Grz *is the logic of the topological sum* $\bigoplus_{n\in\omega}\mathfrak{T}^n_\omega$.

Proof. It is well known that the logic of a topological sum is the intersection of the logics of the summands. Thus, Theorem 5.7 yields that the logic of $\bigoplus_{n\in\omega}\mathfrak{T}^n_\omega$ is $\bigcap_{n\in\omega}\mathsf{Grz}_{n+1} = \mathsf{Grz}$. □

We next give an alternate proof of the well-known completeness result for Grz.

Theorem 5.9. [1, 14] Grz *is the logic of any ordinal space* α *that contains the ordinal space* ω^ω.

Proof. Let $\alpha \geq \omega^\omega$. Since α is scattered, Grz is contained in the logic of α. Suppose that $\mathsf{Grz} \nvdash \varphi$. Because $\mathsf{Grz} = \bigcap_{n\in\omega}\mathsf{Grz}_{n+1}$, there is $n \in \omega$ such that $\mathsf{Grz}_{n+1} \nvdash \varphi$. By Theorem 5.7(1), $\mathfrak{T}^n_\omega$ refutes φ. By Theorem 5.5, $\mathfrak{T}^n_\omega$ is homeomorphic to ω^n+1. Therefore, $\mathfrak{T}^n_\omega$ is homeomorphic to an open subspace of α. Thus, α refutes φ. □

We conclude the section with Table 5 which summarizes our results thus far.

Logic	is the logic of
Grz_{n+1}	$\mathfrak{T}^n_\omega$ $(n\in\omega)$
Grz	$\bigoplus_{n\in\omega}\mathfrak{T}^n_\omega$
$\mathsf{S4.1}$	$\mathfrak{T}_\kappa$ $(2\leq\kappa<\omega)$
$\mathsf{S4}$	$\mathfrak{T}_\omega$, $\mathfrak{T}^\omega_\omega$, $\mathfrak{T}^\infty_\omega$, and $\mathfrak{T}^\infty_\kappa$ $(2\leq\kappa<\omega)$

TABLE 5. Logics arising in the countable branching case.

6. Generalizing the patch topology for $\mathbf{T}_\kappa$

In this section we generalize the patch topology of a spectral space to the σ-patch topology in the setting of trees, specifically for the spectral spaces $\mathbf{T}_\kappa$. Let $\mathscr{B}$ be the least Boolean algebra containing $\mathscr{S} = \{{\uparrow}\sigma \mid \sigma \in T^\omega_\kappa\}$. Then $\mathscr{B}$ is a basis for the patch topology π of $\mathbf{T}_\kappa$.

Definition 6.1. Let κ be nonzero, let $\mathscr{S} = \{\uparrow\sigma \mid \sigma \in T_\kappa^\omega\}$, and let $\mathscr{A}$ be the least σ-algebra containing $\mathscr{S}$. Define the *σ-patch topology* as the topology Π on T_κ that has $\mathscr{A}$ as a basis.

(1) Let $\mathbb{T}_\kappa = (T_\kappa, \Pi)$.
(2) Let $\mathbb{T}_\kappa^n$ be the subspace of $\mathbb{T}_\kappa$ whose underlying set is T_κ^n.
(3) Let $\mathbb{T}_\kappa^\omega$ be the subspace of $\mathbb{T}_\kappa$ whose underlying set is T_κ^ω.
(4) Let $\mathbb{T}_\kappa^\infty$ be the subspace of $\mathbb{T}_\kappa$ whose underlying set is T_κ^∞.

Since $\mathscr{B} \subseteq \mathscr{A}$, it is clear that the σ-patch topology is finer than the patch topology. The next lemma is straightforward, and we skip its proof.

Lemma 6.2. *If $\mathscr{B}$ is a basis for a space X that is closed under countable intersections, then every G_δ-set of X is open.*

Lemma 6.3. *$\mathbb{T}_\kappa$ is a P-space.*

Proof. It is clear that $\mathbb{T}_\kappa$ is a zero-dimensional Hausdorff space, hence a Tychonoff space. Thus, it follows from Lemma 6.2 that $\mathbb{T}_\kappa$ is a P-space. □

Remark 6.4. If $(0 \neq)\kappa \leq \omega$, then $\mathbb{T}_\kappa$ is discrete because $\sigma \in T_\kappa^\infty$ implies $\{\sigma\} = \bigcap_{n\in\omega} \uparrow(\sigma|_n) \in \mathscr{A} \subseteq \Pi$; and $\sigma \in T_\kappa^\omega$ implies $\{\sigma\} = \uparrow\sigma \setminus \bigcup_{\lambda\in\kappa} \uparrow\sigma.\lambda \in \mathscr{A} \subseteq \Pi$. Therefore, every point is isolated, and hence $\mathbb{T}_\kappa$ is discrete. Thus, we must consider uncountable κ.

Remark 6.5. An infinite sequence $\sigma \in T_\kappa^\infty$ has no children and hence $\sigma.\lambda$ is undefined for any $\lambda \in \kappa$. Despite this fact, it is convenient for introducing a useful basis of $\mathbb{T}_\kappa$ to define $\uparrow\sigma.\lambda = \varnothing$ for infinite σ. Then $\{\sigma\} = \uparrow\sigma = \uparrow\sigma \setminus \bigcup_{\lambda\in\Lambda} \uparrow\sigma.\lambda$ for any subset Λ of κ.

The following is an analogue of Lemma 3.10.

Lemma 6.6. *The family*

$$\mathscr{G} = \{\varnothing\} \cup \left\{\uparrow\sigma \setminus \bigcup\nolimits_{\lambda\in\Lambda} \uparrow\sigma.\lambda \mid \sigma \in T_\kappa \text{ and } \Lambda \subseteq \kappa \text{ is countable}\right\}$$

is closed under countable intersections and is a basis for $\mathbb{T}_\kappa$.

Proof. First we show that $\mathscr{G}$ is closed under countable intersections. The empty intersection is $T_\kappa = \uparrow\varepsilon \in \mathscr{G}$, so we consider countable intersections of nonempty families. Let $U_n \in \mathscr{G}$ for $n \in \omega$ and $U := \bigcap_{n\in\omega} U_n$. Since $\varnothing \in \mathscr{G}$, we may assume that $U \neq \varnothing$. For each $n \in \omega$, there are $\sigma_n \in T_\kappa$ and a countable $\Lambda_n \subseteq \kappa$ such that $U_n = \uparrow\sigma_n \setminus \bigcup_{\lambda\in\Lambda_n} \uparrow\sigma_n.\lambda$. We have

$$U = \bigcap_{n\in\omega}\left(\uparrow\sigma_n \setminus \bigcup_{\lambda\in\Lambda_n} \uparrow\sigma_n.\lambda\right) = \left(\bigcap_{n\in\omega}\uparrow\sigma_n\right) \setminus \bigcup_{n\in\omega}\bigcup_{\lambda\in\Lambda_n} \uparrow\sigma_n.\lambda = \left(\bigcap_{n\in\omega}\uparrow\sigma_n\right) \setminus \bigcup_{\lambda\in\Lambda} \uparrow\sigma_n.\lambda$$

where $\Lambda = \bigcup_{n\in\omega} \Lambda_n$. Note that Λ is countable since it is a countable union of countable sets. Because U is nonempty, there is $\sigma \in \bigcap_{n\in\omega} \uparrow\sigma_n$. We have that $\sigma_n \in \downarrow\sigma$ for all $n \in \omega$. Thus, $C := \{\sigma_n \mid n \in \omega\}$ is a chain and hence a directed set. Since $(T_\kappa, \leq)$ is a DCPO (see Remark 3.6), it follows that $\sup C \in T_\kappa$ and $\bigcap_{n\in\omega} \uparrow\sigma_n = \uparrow \sup C$. Therefore,

$$U = \left(\bigcap_{n\in\omega}\uparrow\sigma_n\right) \setminus \bigcup_{\lambda\in\Lambda} \uparrow\sigma_n.\lambda = \uparrow \sup C \setminus \bigcup_{\lambda\in\Lambda} \uparrow\sigma_n.\lambda \in \mathscr{G},$$

showing that $\mathscr{G}$ is closed under countable intersections.

Let Ω be the topology generated by $\mathscr{G}$. We next show that (T_κ, Ω) is a P-space. Since $\mathscr{G}$ is closed under countable intersections, it is sufficient to show that (T_κ, Ω) is a Tychonoff space, for which it is sufficient to see that (T_κ, Ω) is a zero-dimensional Hausdorff space.

To see that (T_κ, Ω) is Hausdorff, let $\sigma, \varsigma \in T_\kappa$ be distinct. Either σ and ς are related or not. If not, then $\uparrow\sigma \in \mathscr{G}$ and $\uparrow\varsigma \in \mathscr{G}$ are disjoint sets containing σ and ς, respectively. Therefore, without loss of generality we may assume that $\sigma < \varsigma$. Let σ' be the child of σ such that $\sigma' \leq \varsigma$. Then $\uparrow\sigma \setminus \uparrow\sigma' \in \mathscr{G}$ and $\uparrow\varsigma \in \mathscr{G}$ are disjoint sets containing σ and ς, respectively. Thus, (T_κ, Ω) is Hausdorff.

To see that (T_κ, Ω) is zero-dimensional, we show that $T_\kappa \setminus U \in \Omega$ for each $U \in \mathscr{G}$. Let $U \in \mathscr{G}$. Without loss of generality we may assume that $T_\kappa \setminus U \neq \varnothing$. Let $\varsigma \in T_\kappa \setminus U$. We show that ς is an interior point of $T_\kappa \setminus U$. We have that $U = \uparrow\sigma \setminus \bigcup_{\lambda \in \Lambda} \uparrow\sigma.\lambda$ for some $\sigma \in T_\kappa$ and countable $\Lambda \subseteq \kappa$. Either $\varsigma \notin \uparrow\sigma$ or $\varsigma \in \bigcup_{\lambda \in \Lambda} \uparrow\sigma.\lambda$. In the latter case, we have that $\uparrow\varsigma \in \mathscr{G}$ and

$$\varsigma \in \uparrow\varsigma \subseteq \bigcup\nolimits_{\lambda \in \Lambda} \uparrow\sigma.\lambda \subseteq T_\kappa \setminus U.$$

In the former case, let ς' be the sequence of greatest length in $\downarrow\sigma \cap \downarrow\varsigma$. Then ς' is a finite sequence distinct from σ, and so there is a unique child σ' of ς' such that $\sigma' \leq \sigma$. We have that $\uparrow\varsigma' \setminus \uparrow\sigma' \in \mathscr{G}$ and

$$\varsigma \in \uparrow\varsigma' \setminus \uparrow\sigma' \subseteq \uparrow\varsigma' \setminus \uparrow\sigma \subseteq T_\kappa \setminus \uparrow\sigma \subseteq T_\kappa \setminus U.$$

Therefore, each point in $T_\kappa \setminus U$ is interior, yielding that $T_\kappa \setminus U \in \Omega$. Thus, each $U \in \mathscr{G}$ is clopen, and so (T_κ, Ω) is zero-dimensional. It follows that (T_κ, Ω) is Tychonoff, and hence a P-space.

Finally, we show that the topologies Π and Ω are equal. For this we first show that $\mathscr{G} \subseteq \mathscr{A}$. Let $U \in \mathscr{G}$. We may assume that $U \neq \varnothing$, so there are $\sigma \in T_\kappa$ and a countable $\Lambda \subseteq \kappa$ such that $U = \uparrow\sigma \setminus \bigcup_{\lambda \in \Lambda} \uparrow\sigma.\lambda$. If σ is infinite, then $U = \uparrow\sigma \setminus \bigcup_{\lambda \in \Lambda} \uparrow\sigma.\lambda = \{\sigma\} = \bigcap_{n \in \omega} \uparrow(\sigma|_n) \in \mathscr{A}$. Suppose σ is finite. Because Λ is countable and each $\sigma.\lambda$ is finite, we have that $\bigcup_{\lambda \in \Lambda} \uparrow\sigma.\lambda \in \mathscr{A}$, giving that $U = \uparrow\sigma \setminus \bigcup_{\lambda \in \Lambda} \uparrow\sigma.\lambda \in \mathscr{A}$. Thus, $\mathscr{G} \subseteq \mathscr{A}$, and hence $\Omega \subseteq \Pi$.

For the reverse inclusion, let $\mathscr{C}$ denote the set of clopen subsets of (T_κ, Ω). Then $\mathscr{S} \subseteq \mathscr{G} \subseteq \mathscr{C}$. Because (T_κ, Ω) is a P-space, $\mathscr{C}$ is a σ-algebra. Therefore, $\mathscr{A} \subseteq \mathscr{C}$, and hence $\Pi \subseteq \Omega$. Thus, $\Pi = \Omega$. Consequently, $\mathscr{G}$ is a basis for $\mathbb{T}_\kappa$. ☐

Lemma 6.7. *Let κ be uncountable. Then $\mathsf{Iso}(\mathbb{T}_\kappa) = T_\kappa^\infty$ and is dense in $\mathbb{T}_\kappa$.*

Proof. If $\sigma \in T_\kappa^\infty$, then $\{\sigma\} = \bigcap_{n \in \omega} \uparrow(\sigma|_n)$, so $T_\kappa^\infty \subseteq \mathsf{Iso}(\mathbb{T}_\kappa)$. Let $\sigma \in T_\kappa^\omega$ and U be an open neighborhood of σ in $\mathbb{T}_\kappa$. By Lemma 6.6, there are $\varsigma \in T_\kappa^\omega$ and a countable $\Lambda \subseteq \kappa$ such that $\sigma \in \uparrow\varsigma \setminus \bigcup_{\lambda \in \Lambda} \uparrow\varsigma.\lambda \subseteq U$. Since $\uparrow\varsigma \setminus \bigcup_{\lambda \in \Lambda} \uparrow\varsigma.\lambda$ is infinite, σ is not an isolated point. Thus, $\mathsf{Iso}(\mathbb{T}_\kappa) = T_\kappa^\infty$. Moreover, because $\uparrow\varsigma \setminus \bigcup_{\lambda \in \Lambda} \uparrow\varsigma.\lambda$ has a nonempty intersection with T_κ^∞, we see that σ is in the closure of T_κ^∞. Since $T_\kappa = T_\kappa^\infty \cup T_\kappa^\omega$, we conclude that T_κ^∞ is dense in $\mathbb{T}_\kappa$. ☐

Using Lemma 6.6, the next lemma is straightforward, and we leave the proof out.

Lemma 6.8. *Let κ be uncountable. Then each of the spaces $\mathbb{T}_\kappa, \mathbb{T}_\kappa^\infty, \mathbb{T}_\kappa^\omega, \mathbb{T}_\kappa^n$, where $n \in \omega$ is nonzero, has cardinality and weight κ.*

Lemma 6.9. *Let $\sigma \in T_\kappa^\omega$ and $U \in \Pi$. If $\sigma \in U$, then $\uparrow\sigma' \subseteq U$ for all but countably many children σ' of σ.*

Proof. It follows from Lemma 6.6 that there are $\varsigma \in T_\kappa^\omega$ and a countable $\Lambda \subseteq \kappa$ such that $\sigma \in \uparrow\varsigma \setminus \bigcup_{\lambda \in \Lambda} \uparrow\varsigma.\lambda \subseteq U$. If $\varsigma < \sigma$, then there is $\lambda_0 \in \kappa \setminus \Lambda$ such that $\varsigma.\lambda_0 \leq \sigma$. Because $\mathcal{T}_\kappa$ is a tree, it follows that $\uparrow\sigma \subseteq \uparrow\varsigma.\lambda_0 \subseteq T_\kappa \setminus \bigcup_{\lambda \in \Lambda} \uparrow\varsigma.\lambda$. Therefore, $\uparrow\sigma' \subseteq \uparrow\sigma \subseteq \uparrow\varsigma \setminus \bigcup_{\lambda \in \Lambda} \uparrow\varsigma.\lambda \subseteq U$ for all children σ' of σ. If $\varsigma = \sigma$, then since Λ is countable, all but countably many children σ' of σ are in $\uparrow\varsigma \setminus \bigcup_{\lambda \in \Lambda} \uparrow\varsigma.\lambda$. Let σ' be a child of σ, and hence of ς, such that $\sigma' \neq \varsigma.\lambda$ for any $\lambda \in \Lambda$. Then $\uparrow\sigma' \subseteq T_\kappa \setminus \bigcup_{\lambda \in \Lambda} \uparrow\varsigma.\lambda$. Thus, $\uparrow\sigma' \subseteq \uparrow\varsigma \setminus \bigcup_{\lambda \in \Lambda} \uparrow\varsigma.\lambda \subseteq U$ for all but countable many children σ' of σ. ☐

Theorem 6.10. *Let κ be uncountable.*

(1) *The space $\mathbb{T}_\kappa$ is a densely discrete non-Lindelöf P-space of cardinality and weight κ.*
(2) *The space $\mathbb{T}_\kappa^\omega$ is a crowded Lindelöf P-space of cardinality and weight κ.*
(3) *For nonzero $n \in \omega$, the space $\mathbb{T}_\kappa^n$ is a scattered Lindelöf P-space of cardinality and weight κ such that the Cantor-Bendixson rank of $\mathbb{T}_\kappa^n$ is $n+1$.*
(4) *The spaces $\mathbb{T}_\kappa$, $\mathbb{T}_\kappa^\omega$, and $\mathbb{T}_\kappa^n$ $(0 \neq n \in \omega)$ are non-metrizable.*

Proof. (1) By Lemma 6.3, $\mathbb{T}_\kappa$ is a P-space; by Lemma 6.7, $\mathbb{T}_\kappa$ is densely discrete; and by Lemma 6.8, $\mathbb{T}_\kappa$ has cardinality and weight κ. It is left to show that $\mathbb{T}_\kappa$ is not Lindelöf. Let $Z = \{\sigma \in T_\kappa \mid \sigma(n) \in \{0,1\} \text{ for all } n < \ell(\sigma)\}$. For finite $\sigma \in Z$, put $U_\sigma = {\uparrow}\sigma \setminus ({\uparrow}\sigma.0 \cup {\uparrow}\sigma.1)$. For infinite $\sigma \in Z$, put $U_\sigma = \{\sigma\}$. Noting that there are uncountably many infinite sequences in Z, it is clear that $\mathscr{C} := \{U_\sigma \mid \sigma \in Z\}$ is an uncountable collection of open subsets of $\mathbb{T}_\kappa$. We show that $\mathscr{C}$ is a pairwise disjoint cover of $\mathbb{T}_\kappa$.

To see that $\mathscr{C}$ is pairwise disjoint, let $\sigma, \varsigma \in Z$ be distinct. If both σ and ς are infinite sequences, then $U_\sigma \cap U_\varsigma = \{\sigma\} \cap \{\varsigma\} = \varnothing$. Suppose one of the two is finite and the other is infinite. Then without loss of generality we may assume that σ is finite and ς is infinite. If $\varsigma \notin {\uparrow}\sigma$, then clearly $U_\sigma \cap U_\varsigma \subseteq {\uparrow}\sigma \cap \{\varsigma\} = \varnothing$. Suppose $\varsigma \in {\uparrow}\sigma$. Since $\varsigma \in Z$, we have that $\varsigma(\ell(\sigma)) \in \{0,1\}$, which yields that $\varsigma \in {\uparrow}\sigma.0 \cup {\uparrow}\sigma.1$. Thus, $U_\sigma \cap U_\varsigma = ({\uparrow}\sigma \setminus ({\uparrow}\sigma.0 \cup {\uparrow}\sigma.1)) \cap \{\varsigma\} = \varnothing$. Finally, suppose that both σ and ς are finite. If σ and ς are unrelated, then $U_\sigma \cap U_\varsigma \subseteq {\uparrow}\sigma \cap {\uparrow}\varsigma = \varnothing$. So, without loss of generality we may assume that $\varsigma < \sigma$. Since $\sigma \in Z$, we have that $\sigma \in {\uparrow}\varsigma.0 \cup {\uparrow}\varsigma.1$. This yields that ${\uparrow}\sigma \subseteq {\uparrow}\varsigma.0 \cup {\uparrow}\varsigma.1$, and hence $U_\sigma \subseteq {\uparrow}\varsigma.0 \cup {\uparrow}\varsigma.1$. Therefore, $U_\sigma \cap U_\varsigma \subseteq ({\uparrow}\varsigma.0 \cup {\uparrow}\varsigma.1) \cap ({\uparrow}\varsigma \setminus ({\uparrow}\varsigma.0 \cup {\uparrow}\varsigma.1)) = \varnothing$. Thus, $\mathscr{C}$ is pairwise disjoint.

To see that $\mathscr{C}$ covers $\mathbb{T}_\kappa$, let $\sigma \in T_\kappa$. If $\sigma \in Z$, then $\sigma \in U_\sigma$. Suppose that $\sigma \in T_\kappa \setminus Z$. Let N be the least element of $\{n \in \omega \mid \sigma(n) \in \kappa \setminus \{0,1\}\}$. Then $\sigma|_N \in Z$ and $\sigma(N) \notin \{0,1\}$. Therefore, $\sigma \notin {\uparrow}(\sigma|_N).0 \cup {\uparrow}(\sigma|_N).1$. Thus, $\sigma \in {\uparrow}(\sigma|_N) \setminus ({\uparrow}(\sigma|_N).0 \cup {\uparrow}(\sigma|_N).1) = U_{\sigma|_N} \in \mathscr{C}$, yielding that $\mathscr{C}$ covers $\mathbb{T}_\kappa$. Being pairwise disjoint, $\mathscr{C}$ has no proper subcover, and hence no countable subcover. Consequently, $\mathbb{T}_\kappa$ is not Lindelöf.

(2) Noting that being a P-space is a hereditary property, by Lemmas 6.3 and 6.8, we only need to verify that $\mathbb{T}_\kappa^\omega$ is a crowded Lindelöf space. That $\mathbb{T}_\kappa^\omega$ is crowded follows from Lemma 6.9. We show that $\mathbb{T}_\kappa^\omega$ is Lindelöf. Let $\mathscr{C}$ be an open cover of $\mathbb{T}_\kappa^\omega$. Then there is $U_\varepsilon \in \mathscr{C}$ such that $\varepsilon \in U_\varepsilon$. Put $\mathscr{C}_0 = \{U_\varepsilon\}$ and

$$C_0 = \left\{\sigma \in T_\kappa^\omega \mid \ell(\sigma) = 1 \text{ and } {\uparrow}\sigma \not\subseteq \bigcup \mathscr{C}_0\right\}.$$

It follows from Lemma 6.9 that C_0 is countable. Clearly $T_\kappa^0 = \{\varepsilon\} \subseteq U_\varepsilon = \bigcup \mathscr{C}_0$. Let $n \in \omega$. Suppose a countable subset $\mathscr{C}_n$ of $\mathscr{C}$ is given such that $T_\kappa^n \subseteq \bigcup \mathscr{C}_n$ and

$$C_n := \left\{\sigma \in T_\kappa^\omega \mid \ell(\sigma) = n+1 \text{ and } {\uparrow}\sigma \not\subseteq \bigcup \mathscr{C}_n\right\}$$

is countable. Let $\sigma \in C_n$. Then there is $U_\sigma \in \mathscr{C}$ such that $\sigma \in U_\sigma$, and $\mathscr{C}_{n+1} := \mathscr{C}_n \cup \{U_\sigma \mid \sigma \in C_n\}$ is countable. Clearly $T_\kappa^n \subseteq \bigcup \mathscr{C}_n \subseteq \bigcup \mathscr{C}_{n+1}$. Let $\sigma \in T_\kappa^{n+1}$ be of length $n+1$. If ${\uparrow}\sigma \subseteq \bigcup \mathscr{C}_n$, then ${\uparrow}\sigma \subseteq \bigcup \mathscr{C}_{n+1}$. So suppose that ${\uparrow}\sigma \not\subseteq \bigcup \mathscr{C}_n$. Then $\sigma \in C_n$, hence $\sigma \in U_\sigma \subseteq \bigcup \mathscr{C}_{n+1}$, and we have that $T_\kappa^{n+1} \subseteq \bigcup \mathscr{C}_{n+1}$.

Consider $\mathscr{C}' := \bigcup_{n \in \omega} \mathscr{C}_n \subseteq \mathscr{C}$. Clearly $\mathscr{C}'$ is countable and $\mathscr{C}'$ covers T_κ^ω since

$$T_\kappa^\omega = \bigcup\nolimits_{n \in \omega} T_\kappa^n \subseteq \bigcup\nolimits_{n \in \omega} \bigcup \mathscr{C}_n \subseteq \bigcup\nolimits_{n \in \omega} \bigcup \mathscr{C}' \subseteq \bigcup \mathscr{C}'.$$

Thus, $\mathbb{T}_\kappa^\omega$ is Lindelöf.

(3) Let $n \in \omega$ be nonzero. As in (2), in light of Lemmas 6.3 and 6.8, we only need to verify that $\mathbb{T}_\kappa^n$ is Lindelöf, scattered, and of Cantor-Bendixson rank $n+1$. First note that $\mathbb{T}_\kappa^n$ is a closed subspace of $\mathbb{T}_\kappa^\omega$ since $T_\kappa^\omega \setminus T_\kappa^n = \bigcup\{{\uparrow}\sigma \cap T_\kappa^\omega \mid \sigma \in T_\kappa^\omega \text{ with } \ell(\sigma) = n+1\}$ is open in $\mathbb{T}_\kappa^\omega$. Therefore, $\mathbb{T}_\kappa^n$ is Lindelöf by [18, Thm. 3.8.4]. Clearly each $\sigma \in \mathbb{T}_\kappa^n$ of length n

is isolated since $\{\sigma\} = {\uparrow}\sigma \cap T^n_\kappa$. By Lemma 6.9, each $\sigma \in \mathbb{T}^n_\kappa$ of length $< n$ is not isolated. Thus, $\mathsf{Iso}(\mathbb{T}^n_\kappa) = T^n_\kappa \setminus T^{n-1}_\kappa$. Consequently, $\mathsf{d}(T^n_\kappa) = T^{n-1}_\kappa$, and it is easily checked that $\mathsf{d}^k(T^n_\kappa) = T^{n-k}_\kappa$ for $0 \le k \le n$ and $\mathsf{d}^{n+1}(T^n_\kappa) = \varnothing$. Therefore, $\mathbb{T}^n_\kappa$ is scattered, and is of Cantor-Bendixson rank $n+1$.

(4) Note that every metrizable P-space is discrete (because each singleton is a G_δ-set). Since the spaces $\mathbb{T}_\kappa$, $\mathbb{T}^\omega_\kappa$, and $\mathbb{T}^n_\kappa$ $(0 \neq n \in \omega)$ are non-discrete P-spaces, it follows that each is non-metrizable. □

Recall that the *one-point Lindelöfication* of an uncountable discrete space D is obtained by adding a point to D whose neighborhoods have countable complement in D. Formally we topologize $D \cup \{p\}$ for some $p \notin D$ by defining U to be open if either $U \subseteq D$ or $D \setminus U$ is countable. The one-point Lindelöfication of an uncountable discrete space is analogous to the one-point compactification of an infinite discrete space, and is realized by replacing the 'finite complement' clause in the definition of the latter with the 'countable complement' clause of the former.

Remark 6.11. In comparison to Remark 5.6, we point out the following:

(1) The space $\mathbb{T}^1_\kappa$ is the one-point Lindelöfication of the discrete space κ, whereas $\mathfrak{T}^1_\omega$ is the one-point compactification of the discrete space ω.
(2) Analogously to $\mathfrak{T}^{n+1}_\omega$, the space $\mathbb{T}^{n+1}_\kappa$ is obtained as an adjunction space from $\mathbb{T}^n_\kappa$ and κ many copies of $\mathbb{T}^1_\kappa$ by gluing the limit points (roots) of the copies of $\mathbb{T}^1_\kappa$ to the isolated points (leafs) of $\mathbb{T}^n_\kappa$.

Remark 6.12. Comparing $\mathfrak{T}_\omega$ and $\mathbb{T}_\kappa$, both are zero-dimensional Hausdorff spaces. On the other hand, $\mathfrak{T}_\omega$ is crowded, compact, and (completely) metrizable, while $\mathbb{T}_\kappa$ is densely discrete and is neither Lindelöf nor metrizable. The infinite sequences in both spaces form a dense subset, whereas the finite sequences are dense in $\mathfrak{T}_\omega$ while the finite sequences are closed in $\mathbb{T}_\kappa$.

7. Some logics arising from the σ-patch topology on trees

This section parallels the results of Sections 4 and 5, but the topological completeness results we will obtain are with respect to non-metrizable spaces, by considering the σ-patch topology on trees arising from uncountable κ.

The next theorem is a modification of Theorem 4.12, so we only sketch the proof.

Theorem 7.1. *Suppose that κ is uncountable.*

(1) *The Alexandroff space $\mathcal{T}^\omega_\omega$ is an interior image of $\mathbb{T}^\omega_\kappa$.*
(2) *For $n \in \omega$, the Alexandroff space $\mathcal{T}^n_\omega$ is an interior image of $\mathbb{T}^n_\kappa$.*

Proof. Let $\{K_n \mid n \in \omega\}$ be a partition of κ such that each K_n is uncountable. Recursively define $f : T^\omega_\kappa \to T^\omega_\omega$ by $f(\varepsilon) = \varepsilon$ and $f(\sigma.\alpha) = f(\sigma).n$ whenever $f(\sigma)$ is defined and $\alpha \in K_n$. Then f is a well-defined onto mapping that is continuous because the Alexandroff topology on T^ω_κ is coarser than the σ-patch topology on T^ω_κ. That f is open follows from Lemma 6.6 and the equality $f\left(({\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda) \cap T^\omega_\kappa\right) = {\uparrow}f(\sigma) \cap T^\omega_\omega$ for any $\sigma \in T^\omega_\kappa$ and countable $\Lambda \subseteq \kappa$, which in turn depends on each K_n being uncountable and Λ being countable. Now (2) follows from Lemma 4.11 since $f^{-1}(T^n_\omega) = T^n_\kappa$. □

Remark 7.2. On the other hand, the space $\mathfrak{T}^\omega_\omega$ is *not* an interior image of $\mathbb{T}^\omega_\kappa$ via the map f defined in the proof of Theorem 7.1. Indeed, $f^{-1}({\uparrow}\varepsilon \setminus {\uparrow}\varepsilon.0) = {\uparrow}\varepsilon \setminus \bigcup_{\alpha\in K_0} {\uparrow}\varepsilon.\alpha$ is not open in $\mathbb{T}^\omega_\kappa$ since K_0 is uncountable (see Lemma 6.9).

Theorem 7.3. *Let κ be uncountable.*

(1) *For $n \in \omega$, the logic of* $\mathbb{T}^n_\kappa$ *is* Grz_{n+1}.
(2) *The logic of* $\bigoplus_{n\in\omega} \mathbb{T}^n_\kappa$ *is* Grz.

Proof. (1) By Theorem 6.10(3), $\mathbb{T}^n_\kappa$ is a scattered space such that $r(\mathbb{T}^n_\kappa) = n + 1$. Thus, by [7, Lem. 3.6], $\mathsf{Grz}_{n+1} \subseteq \mathsf{Log}(\mathbb{T}^n_\kappa)$. For the reverse inclusion, suppose that $\mathsf{Grz}_{n+1} \nvdash \varphi$. Then $\mathcal{T}^n_\omega$ refutes φ. As $\mathcal{T}^n_\omega$ is an interior image of $\mathbb{T}^n_\kappa$ (see Theorem 7.1(2)), $\mathbb{T}^n_\kappa$ refutes φ. Thus, $\mathsf{Log}(\mathbb{T}^n_\kappa) = \mathsf{Grz}_{n+1}$.

(2) By (1), we have $\mathsf{Log}\left(\bigoplus_{n\in\omega} \mathbb{T}^n_\kappa\right) = \bigcap_{n\in\omega} \mathsf{Log}(\mathbb{T}^n_\kappa) = \bigcap_{n\in\omega} \mathsf{Grz}_{n+1} = \mathsf{Grz}$. □

The next theorem is a modification of Theorem 4.9. But since the isolated points of $\mathbb{T}_\kappa$ are the infinite sequences, whereas the isolated points of $\mathfrak{T}_\lambda$ are the finite sequences when λ is finite, the construction is simpler because there is no need to adjust the images of finite sequences.

Theorem 7.4. *Let κ be uncountable and $\mathfrak{F} = (W, R)$ a finite top-thin-quasi-tree obtained from $\mathfrak{G} = (V, Q)$. Then $\mathfrak{F}$ is an interior image of $\mathbb{T}_\kappa$.*

Proof. Let $\{K_n \mid n \in \omega\}$ be a partition of κ such that each K_n is uncountable. For each $w \in V$, choose and fix an enumeration $\{w_n \mid n < n_w\}$ of $Q(w)$. For each cluster C in $\mathfrak{G}$, choose and fix $m_C \in \max(\mathfrak{F})$ such that wRm_C for all $w \in C$. Since $\mathfrak{F}$ is top-thin, for each maximal cluster C in $\mathfrak{G}$ there is a unique such m_C. Let r be a root of $\mathfrak{G}$, and hence a root of $\mathfrak{F}$.

Recursively define $f : T_\kappa \to W$ as follows. Set $f(\varepsilon) = r$. Assume $\sigma \in T^\omega_\kappa$ and $f(\sigma) = w \in V$. Set $f(\sigma.\alpha) = w_{n \bmod n_w}$ whenever $\alpha \in K_n$. Clearly $f(\sigma.\alpha) \in V$. Let $\sigma \in T^\infty_\kappa$. Then $\{f(\sigma|_n) \mid n \in \omega\}$ is a Q-increasing sequence in $\mathfrak{G}$. Since $\mathfrak{G}$ is finite, there are $N \in \omega$ and a cluster C in $\mathfrak{G}$ such that $f(\sigma|_n) \in C$ for all $n \geq N$. Set $f(\sigma) = m_C$. It follows by transfinite induction on the length of $\sigma \in T_\kappa$ that f is well defined.

Claim 7.5. *If $\sigma \leq \varsigma$, then $f(\sigma)Rf(\varsigma)$.*

Proof. Suppose $\sigma \in T^\infty_\kappa$. Then $\sigma = \varsigma$, and so $f(\sigma) = f(\varsigma)$, which yields $f(\sigma)Rf(\varsigma)$ since R is reflexive. So assume $\sigma \in T^\omega_\kappa$ and $f(\sigma) = w \in V$. Then by the definition of f, for each child $\sigma.\alpha$ of σ where $\alpha \in K_n$, we have that $f(\sigma.\alpha) = w_{n \bmod n_w} \in Q(w) \subseteq R(w) = R(f(\sigma))$. Thus, for $\varsigma \in T^\omega_\kappa$, it follows by induction that $f(\sigma)Rf(\varsigma)$ since both $\leq$ and R are transitive. Suppose $\varsigma \in T^\infty_\kappa$. There are $N \in \omega$ and a cluster C in $\mathfrak{G}$ such that $f(\varsigma|_n) \in C$ for all $n \geq N$ and $f(\varsigma) = m_C$. Since $\sigma \leq \varsigma$, we have that $f(\sigma) \in \{f(\varsigma|_n) \mid n \in \omega\}$. Because ${\downarrow}\varsigma$ is a chain, $\sigma \leq \varsigma|_N$ or $\varsigma|_N \leq \sigma$. If $\sigma \leq \varsigma|_N$, then $f(\sigma)Rf(\varsigma|_N)Rm_C = f(\varsigma)$. If $\varsigma|_N \leq \sigma$, then $f(\sigma) \in C$, and so $f(\sigma)Rm_C = f(\varsigma)$. In either case, $f(\sigma)Rf(\varsigma)$. □

It follows from Claim 7.5 that f viewed as a mapping from the Alexandroff space $\mathcal{T}_\kappa$ to $\mathfrak{F}$ is continuous. Since the σ-patch topology of $\mathbb{T}_\kappa$ is finer than the Alexandroff topology of $\mathcal{T}_\kappa$, we have that $f : \mathbb{T}_\kappa \to \mathfrak{F}$ is continuous.

Claim 7.6. *Let $\sigma \in T_\kappa$, $w \in W$, and $\Lambda \subseteq \kappa$ be countable. If $f(\sigma)Rw$, then there is $\varsigma \in {\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda$ such that $f(\varsigma) = w$.*

Proof. Suppose $f(\sigma) \in \max(\mathfrak{F})$. Then $f(\sigma) = w$ and we can take $\varsigma = \sigma$. Assume $f(\sigma) \notin \max(\mathfrak{F})$. Then $f(\sigma) \in V$, say $f(\sigma) = v$. Either $w \in V$ or $w \in W \setminus V$. First suppose $w \in V$. Then $w \in Q(f(\sigma)) = Q(v)$ and there is $n < n_v$ such that $w = v_n$ in the enumeration of $Q(v)$. Because K_n is uncountable and Λ is countable, we may consider $\alpha \in K_n \setminus \Lambda$ and $\varsigma := \sigma.\alpha$. Then $\varsigma \in {\uparrow}\sigma \setminus \bigcup_{\lambda\in\Lambda} {\uparrow}\sigma.\lambda$ and $f(\varsigma) = f(\sigma.\alpha) = v_{n \bmod n_v} = v_n = w$ as desired.

Next suppose $w \in W \setminus V = \max(\mathfrak{F})$. Since $\mathfrak{F}$ is a finite top-thin-quasi-tree, there is a maximal cluster C in $\mathfrak{G}$ such that $R(C) = C \cup \{w\}$, and hence $m_C = w$. Let $w' \in C$. Because C is a maximal cluster in $\mathfrak{G}$, we have that w' is quasi-maximal in $\mathfrak{G}$. Note that

$v = f(\sigma)$ and w' are in the quasi-chain $R^{-1}(w)$. Therefore, either vRw' or $w'Rv$, giving that vQw' or $w'Qv$ since $v, w' \in V$. In the latter case, since w' is quasi-maximal in $\mathfrak{G}$, we get that vQw'. Thus, in either case we have $w' \in Q(v)$. Since $w' \in V$, as shown in the preceding paragraph, there is a child ς' of σ contained in $\uparrow\sigma \setminus \bigcup_{\lambda\in\Lambda} \uparrow\sigma.\lambda$ such that $f(\varsigma') = w'$. We have $\uparrow\varsigma' \subseteq \uparrow\sigma \setminus \bigcup_{\lambda\in\Lambda} \uparrow\sigma.\lambda$ since $\mathcal{T}_\kappa$ is a tree. Let $\varsigma \in \uparrow\varsigma'$ be infinite. Then $\varsigma \in \uparrow\sigma \setminus \bigcup_{\lambda\in\Lambda} \uparrow\sigma.\lambda$. It follows from Claim 7.5 that for $n \geq \ell(\varsigma')$ we have $w' = f(\varsigma')Rf(\varsigma|_n)$. By definition of f, $f(\varsigma|_n) \in V$ for any $n \in \omega$. Therefore, $f(\varsigma|_n) \in C$ whenever $n \geq \ell(\varsigma')$. Because ς is infinite, the definition of f yields that $f(\varsigma) = m_C = w$. ☐

To see that $f : \mathbb{T}_\kappa \to \mathfrak{F}$ is open, we consider basic open sets delivered by Lemma 6.6. Let $U = \uparrow\sigma \setminus \bigcup_{\lambda\in\Lambda} \uparrow\sigma.\lambda$ for some $\sigma \in T_\kappa$ and countable $\Lambda \subseteq \kappa$. By Claim 7.5, $f(U) \subseteq R(f(\sigma))$; and by Claim 7.6, $R(f(\sigma)) \subseteq f(U)$. Therefore, $f(U) = R(f(\sigma))$, giving that f is open. It now follows easily that f is onto since $f(T_\kappa) = f(\uparrow\varepsilon) = R(f(\varepsilon)) = R(r) = W$. Thus, $\mathfrak{F}$ is an interior image of $\mathbb{T}_\kappa$. ☐

Corollary 7.7. *Each finite quasi-tree $\mathfrak{G}$ is an interior image of $\mathbb{T}^\omega_\kappa$.*

Proof. Form a top-thin-quasi-tree $\mathfrak{F}$ from $\mathfrak{G} = (V, Q)$ (see Figure 2). By Theorem 7.4, there is an onto interior map $f : \mathbb{T}_\kappa \to \mathfrak{F}$. The proof of Theorem 7.4 implies that $f^{-1}(V) = T^\omega_\kappa$. Thus, by Lemma 4.11, the restriction of f to T^ω_κ is an interior mapping of $\mathbb{T}^\omega_\kappa$ onto $\mathfrak{G}$. ☐

Theorem 7.8. *If κ is uncountable, then the logic of $\mathbb{T}_\kappa$ is* $\mathsf{S4.1}$.

Proof. By Theorem 6.10(1), $\mathbb{T}_\kappa$ is a densely discrete space, so $\mathsf{S4.1} \subseteq \mathsf{Log}(\mathbb{T}_\kappa)$. For the reverse inclusion, suppose that $\mathsf{S4.1} \nvdash \varphi$. Then φ is refuted on a finite top-thin-quasi-tree $\mathfrak{F}$. By Theorem 7.4, $\mathfrak{F}$ is an interior image of $\mathbb{T}_\kappa$. Thus, $\mathbb{T}_\kappa \not\models \varphi$, and hence $\mathsf{Log}(\mathbb{T}_\kappa) = \mathsf{S4.1}$. ☐

Theorem 7.9. *If κ is uncountable, then the logic of $\mathbb{T}^\omega_\kappa$ is* $\mathsf{S4}$.

Proof. Clearly, $\mathsf{S4} \subseteq \mathsf{Log}(\mathbb{T}^\omega_\kappa)$. Suppose $\mathsf{S4} \nvdash \varphi$. Then φ is refuted on a finite quasi-tree $\mathfrak{F}$. By Corollary 7.7, $\mathfrak{F}$ is an interior image of $\mathbb{T}^\omega_\kappa$. Thus, $\mathbb{T}^\omega_\kappa \not\models \varphi$, giving that $\mathsf{Log}(\mathbb{T}^\omega_\kappa) = \mathsf{S4}$. ☐

Table 6 summarizes the results of this section for uncountable κ.

Logic	is the logic of
Grz_{n+1}	$\mathbb{T}^n_\kappa$ $(n \in \omega)$
Grz	$\bigoplus_{n\in\omega} \mathbb{T}^n_\kappa$
$\mathsf{S4.1}$	$\mathbb{T}_\kappa$
$\mathsf{S4}$	$\mathbb{T}^\omega_\kappa$

TABLE 6. Logics arising in the uncountable branching case.

Remark 7.10. All logics in Table 6 are realized via a single uncountable κ. On the other hand, when κ is countable, it is necessary to vary κ to realize all these logics; see Table 5.

8. Embeddings of trees into ED-spaces and corresponding logics

In this section, we construct Tychonoff ED-spaces that give rise to the following logics:

$$\mathsf{S4.2} \subset \mathsf{S4.1.2} \subset \mathsf{Grz.2} \subset \cdots \subset \mathsf{Grz.2}_3 \subset \mathsf{Grz.2}_2.$$

It is shown in [10] that $\mathsf{S4.1.2}$ is the logic of the Čech-Stone compactification of the discrete space ω, and this result is utilized in [11] to prove that $\mathsf{S4.2}$ is the logic of the Gleason cover of $[0, 1]$. However, the proofs require a set-theoretic assumption beyond ZFC. In contrast, all our proofs are within ZFC. Our basic construction is to embed, for any uncountable cardinal κ, the space $\mathbb{T}_\kappa^\omega$ into the Čech-Stone compactification $\beta(2^\kappa)$ of the discrete space 2^κ. This will yield the desired topological completeness for all the logics in the list except $\mathsf{S4.2}$. For $\mathsf{S4.2}$ we replace $\beta(2^\kappa)$ by the Gleason cover of a large enough power of $[0,1]$.

Let κ be an uncountable cardinal. We identify 2^κ with its image in $\beta(2^\kappa)$ and note that $\mathsf{Iso}(\beta(2^\kappa)) = 2^\kappa$, which is dense in $\beta(2^\kappa)$. It is an unpublished theorem of van Douwen that every P-space of weight κ can be embedded into $\beta(2^\kappa)$. For a proof and a generalization of van Douwen's theorem see [17]. Recall from Theorem 6.10(2) that $\mathbb{T}_\kappa^\omega$ is a P-space of weight κ. Therefore, by van Douwen's theorem, there is an embedding $\iota : \mathbb{T}_\kappa^\omega \to \beta(2^\kappa)$. Since $\mathbb{T}_\kappa^\omega$ is crowded, $\iota(\mathbb{T}_\kappa^\omega) \subseteq \beta(2^\kappa) \setminus 2^\kappa$. Figure 4 depicts the embedding ι.

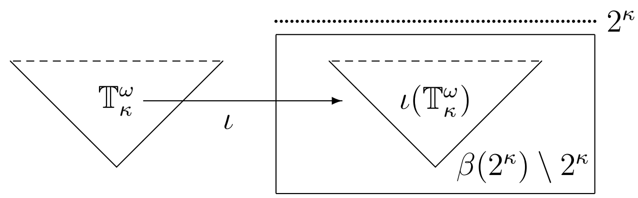

FIGURE 4. The embedding $\iota : \mathbb{T}_\kappa^\omega \to \beta(2^\kappa)$.

Definition 8.1. Identify $\mathbb{T}_\kappa^\omega$ with its image $\iota(\mathbb{T}_\kappa^\omega)$. Define the following subspaces of $\beta(2^\kappa)$:

(1) $X_\kappa^\omega = \mathbb{T}_\kappa^\omega \cup 2^\kappa$;
(2) $X_\kappa^n = \mathbb{T}_\kappa^n \cup 2^\kappa$ $(n \in \omega)$.

Lemma 8.2.

(1) *X_κ^ω is a non-scattered densely discrete ED-space.*
(2) *For $n \in \omega$, X_κ^n is a scattered ED-space of Cantor-Bendixson rank $n+2$.*

Proof. We first show that each of the spaces X_κ^n and X_κ^ω is ED and densely discrete. Let $\alpha \le \omega$. Since 2^κ is dense in $\beta(2^\kappa)$ and $2^\kappa \subseteq X_\kappa^\alpha$, we have that X_κ^α is dense in $\beta(2^\kappa)$. Being a dense subspace of an ED-space, X_κ^α is ED. Because $\mathbb{T}_\kappa^n \subseteq \mathbb{T}_\kappa^\omega \subseteq \beta(2^\kappa) \setminus 2^\kappa$, we have that $\mathsf{Iso}(X_\kappa^\alpha) = 2^\kappa$. Thus, $\mathsf{Iso}(X_\kappa^\alpha)$ is dense in X_κ^α, yielding that X_κ^α is densely discrete.

To complete the proof of (1), we need only observe that X_κ^ω is not scattered. Indeed,

$$\mathsf{d}(X_\kappa^\omega) = X_\kappa^\omega \setminus \mathsf{Iso}(X_\kappa^\omega) = X_\kappa^\omega \setminus 2^\kappa = \mathbb{T}_\kappa^\omega$$

is crowded, so X_κ^ω is not scattered.

Turning to (2), since $\mathsf{d}(X_\kappa^n) = \mathbb{T}_\kappa^n$, $\mathsf{d}^n(\mathbb{T}_\kappa^n) = \{\varepsilon\}$, and $\mathsf{d}^{n+1}(\mathbb{T}_\kappa^n) = \varnothing$, we conclude that $\mathsf{d}^{n+1}(X_\kappa^n) = \{\varepsilon\}$ and $\mathsf{d}^{n+2}(X_\kappa^n) = \varnothing$. Thus, $r(X_\kappa^n) = n+2$. □

Lemma 8.3. *Let $\mathfrak{F} = (W, R)$ be an $\mathsf{S4}$-frame with a unique maximal cluster C consisting of $n \in \omega \setminus \{0\}$ points such that $W \ne C$, and $\mathfrak{G}$ the subframe of $\mathfrak{F}$ whose underlying set is $W \setminus C$. Let X be a space and Y a closed nowhere dense subspace of X such that $X \setminus Y$ is n-resolvable. If $\mathfrak{G}$ is an interior image of Y, then $\mathfrak{F}$ is an interior image of X.*

Proof. Suppose $\mathfrak{G}$ is an interior image of Y, say via $g : Y \to W \setminus C$. Since $X \setminus Y$ is n-resolvable, by [4, Lem. 5.9], there is an onto interior mapping $h : X \setminus Y \to C$. We extend g to $f : X \to W$ by setting $f(x) = h(x)$ for $x \in X \setminus Y$. Then f is a well-defined onto map.

To see that f is continuous, let $F \subseteq W$ be closed in $\mathfrak{F}$, so $F = R^{-1}(F)$. If $F \cap C = \varnothing$, then $f^{-1}(F) = g^{-1}(F)$ is closed in Y since g is continuous. Thus, $f^{-1}(F)$ is closed in X as Y is closed in X. If $F \cap C \ne \varnothing$, then $F = W$ because $F = R^{-1}(F)$ and C is the unique maximal cluster of $\mathfrak{F}$. Therefore, $f^{-1}(F) = X$ is closed in X. Consequently, f is continuous.

To see that f is open, let U be a nonempty open subset of X. Because Y is nowhere dense, $X \setminus Y$ is dense in X. Thus, $U \setminus Y = U \cap (X \setminus Y)$ is a nonempty open subset of $X \setminus Y$. Since $U \cap Y$ is open in Y, the image $g(U \cap Y)$ is open in $\mathfrak{G}$. Therefore,

$$\begin{aligned} f(U) &= f((U \cap Y) \cup (U \setminus Y)) = f(U \cap Y) \cup f(U \setminus Y) \\ &= g(U \cap Y) \cup h(U \setminus Y) = g(U \cap Y) \cup C \end{aligned}$$

is open in $\mathfrak{F}$. Thus, f is open, and hence interior. □

Theorem 8.4. *Let $\mathfrak{F} = (W, R)$ be a finite rooted* $\mathsf{S4}$*-frame with a unique maximal cluster consisting of a single point, say m. Then $\mathfrak{F}$ is an interior image of X_κ^ω.*

Proof. If $W = \{m\}$, then the result is clear. Suppose that $W \neq \{m\}$. Let $\mathfrak{G}$ be the subframe of $\mathfrak{F}$ whose underlying set is $W \setminus \{m\}$. By Lemma 8.2(1), X_κ^ω is densely discrete. Therefore, $\mathbb{T}_\kappa^\omega = \mathsf{d}(X_\kappa^\omega)$ is a closed nowhere dense subspace of X_κ^ω. Now, $\mathfrak{G}$ is an interior image of some finite quasi-tree (see, e.g., [9, Lem. 5]), which by Corollary 7.7, is an interior image of $\mathbb{T}_\kappa^\omega$. Thus, there is an onto interior mapping $g : \mathbb{T}_\kappa^\omega \to \mathfrak{G}$. Since $X_\kappa^\omega \setminus \mathbb{T}_\kappa^\omega \neq \varnothing$ (and any nonempty space is 1-resolvable), we may apply Lemma 8.3 to yield that $\mathfrak{F}$ is an interior image of X_κ^ω. □

Theorem 8.5. *The logic of X_κ^ω is* $\mathsf{S4.1.2}$*.*

Proof. By Lemma 8.2(1), X_κ^ω is a densely discrete ED-space. Therefore, $\mathsf{S4.1.2} \subseteq \mathsf{Log}(X_\kappa^\omega)$. Suppose that $\mathsf{S4.1.2} \not\vdash \varphi$. Then φ is refuted on a finite rooted $\mathsf{S4}$-frame $\mathfrak{F}$ with a unique maximal point. By Theorem 8.4, $\mathfrak{F}$ is an interior image of X_κ^ω. Thus, X_κ^ω refutes φ, and hence $\mathsf{Log}(X_\kappa^\omega) = \mathsf{S4.1.2}$. □

Remark 8.6.

(1) If κ is the least uncountable ordinal ω_1, then there is an embedding of $\mathbb{T}_{\omega_1}^\omega$ into the remainder of $\beta(\omega)$. Let X be the subspace of $\beta(\omega)$ obtained as the union of the image of $\mathbb{T}_{\omega_1}^\omega$ under the aforementioned embedding and ω. Analogous to the above proofs, the logic of X is $\mathsf{S4.1.2}$.

(2) Using set-theoretic assumptions beyond ZFC, it was shown in [10] that $\mathsf{S4.1.2}$ is the logic of $\beta(\omega)$. As follows from (1), we can obtain completeness of $\mathsf{S4.1.2}$ within ZFC for a subspace of $\beta(\omega)$. It remains an open problem whether it can be proved within ZFC that $\mathsf{S4.1.2} = \mathsf{Log}(\beta(\omega))$.

For an $\mathsf{S4}$-frame $\mathfrak{F}$, let $\widehat{\mathfrak{F}}$ be the frame obtained from $\mathfrak{F}$ by adding a new unique maximal point as in Figure 5.

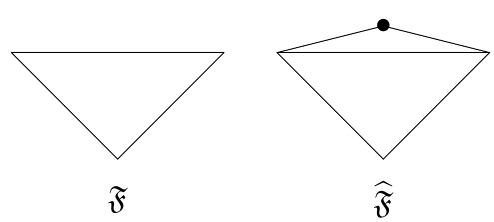

FIGURE 5. Obtaining $\widehat{\mathfrak{F}}$ from $\mathfrak{F}$.

As we pointed out before Theorem 5.7, each finite tree $\mathfrak{F}$ of depth $\leq n+1$ is a p-morphic image of $\mathcal{T}_\omega^n$. This p-morphism clearly extends to a p-morphism from $\widehat{\mathcal{T}_\omega^n}$ onto $\widehat{\mathfrak{F}}$. Since $\mathsf{Grz.2}_{n+2}$ is the logic of the class of frames $\widehat{\mathfrak{F}}$ where $\mathfrak{F}$ is a finite tree of depth $\leq n+1$, we conclude that $\mathsf{Grz.2}_{n+2}$ is the logic of $\widehat{\mathcal{T}_\omega^n}$.

Theorem 8.7. *For each $n \in \omega$, the poset $\widehat{\mathcal{T}_\omega^n}$ is an interior image of X_κ^n.*

Proof. We have that $\mathbb{T}_\kappa^n = \mathsf{d}(X_\kappa^n)$ is nowhere dense since X_κ^n is scattered by Lemma 8.2(2). By Theorem 7.1(2), there is an onto interior mapping $g : \mathbb{T}_\kappa^n \to \mathcal{T}_\omega^n$. Now apply Lemma 8.3. ☐

Theorem 8.8. *For each $n \in \omega$, the logic of X_κ^n is $\mathsf{Grz.2}_{n+2}$.*

Proof. By Lemma 8.2(2), X_κ^n is a scattered ED-space of Cantor-Bendixson rank $n+2$. Therefore, $\mathsf{Grz.2}_{n+2} \subseteq \mathsf{Log}(X_\kappa^n)$. Suppose that $\mathsf{Grz.2}_{n+2} \nvdash \varphi$. Then φ is refuted on $\widehat{\mathcal{T}_\omega^n}$. By Theorem 8.7, $\widehat{\mathcal{T}_\omega^n}$ is an interior image of X_κ^n. Thus, X_κ^n refutes φ, and hence $\mathsf{Log}(X_\kappa^n) = \mathsf{Grz.2}_{n+2}$. ☐

Corollary 8.9. *The logic of $\bigoplus_{n\in\omega} X_\kappa^n$ is $\mathsf{Grz.2}$.*

Proof. By Theorem 8.8, we have

$$\mathsf{Log}\left(\bigoplus_{n\in\omega} X_\kappa^n\right) = \bigcap_{n\in\omega} \mathsf{Log}(X_\kappa^n) = \bigcap_{n\in\omega} \mathsf{Grz.2}_{n+2} = \mathsf{Grz.2}.$$

☐

For each uncountable cardinal κ, we now construct, within ZFC, a space X_κ whose logic is $\mathsf{S4.2}$. For this, let $X = [0,1]^{2^{2^\kappa}}$, let E be the Gleason cover of X, and let $\pi : E \to X$ be the associated irreducible map (see, e.g., [20, Ch. III.3]).

Lemma 8.10. *$\beta(2^\kappa)$ is homeomorphic to a closed nowhere dense subspace of E.*

Proof. By [18, Thm. 3.6.11], $\beta(2^\kappa)$ has weight 2^{2^κ}. Therefore, by [18, Thm. 3.2.5], $\beta(2^\kappa)$ is homeomorphic to a closed subspace of X. Let D be the discrete subset of X corresponding to 2^κ. Then $\beta(2^\kappa)$ is homeomorphic to $\mathsf{c}_X(D)$, the closure of D in X. Let $F \subseteq E$ be such that $F \cap \pi^{-1}(x)$ is a singleton for each $x \in D$. Clearly $\pi(F) = D$; and since π is continuous and D is discrete in X, F is discrete in E. Because π is a closed map, $\pi(\mathsf{c}_E F) = \mathsf{c}_X \pi(E) = \mathsf{c}_X(D)$. Therefore, the following diagram commutes:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \mathsf{c}_E F \arrow[d] \\ D \arrow[r] & \mathsf{c}_X D \end{tikzcd}$$

Since the discrete spaces F and D are homeomorphic and $\mathsf{c}_X(D)$ is homeomorphic to $\beta(2^\kappa)$, we conclude that $\mathsf{c}_E(F)$ is homeomorphic to $\beta(2^\kappa)$. Clearly $\mathsf{c}_E(F)$ is a closed subspace of E. Because X is crowded, E is crowded. Thus, $\mathsf{c}_E(F)$ is nowhere dense in E. ☐

For convenience, we identify $\beta(2^\kappa)$ with $\mathsf{c}_E(F)$. Hence, up to homeomorphism, $\mathbb{T}_\kappa^\omega$ is a nowhere dense subspace of E; see Figure 6.

Definition 8.11. Let X_κ denote the subspace $\mathbb{T}_\kappa^\omega \cup (E \setminus \beta(2^\kappa))$ of E.

Lemma 8.12.
1. *The subset $E \setminus \beta(2^\kappa)$ is open and dense in X_κ.*
2. *The subspace $\mathbb{T}_\kappa^\omega$ is closed and nowhere dense in X_κ.*
3. *The space X_κ is a crowded ED-space.*
4. *For each $n \in \omega$ with $n \geq 1$, the subspace $E \setminus \beta(2^\kappa)$ is n-resolvable.*

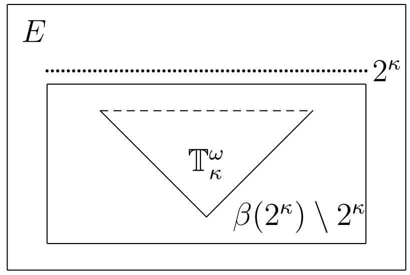

FIGURE 6. Realizing $\mathbb{T}_\kappa^\omega$ as a nowhere dense subspace of E.

Proof. (1) Since $\beta(2^\kappa)$ is closed and nowhere dense in E, it follows that $E \setminus \beta(2^\kappa)$ is open and dense in X_κ.

(2) This is immediate from (1).

(3) It follows from (1) that X_κ is a dense subspace of E. This, together with E being an ED-space, yields that X_κ is an ED-space. Since E is crowded and $E \setminus \beta(2^\kappa)$ is open in E, no point in $E \setminus \beta(2^\kappa)$ is isolated (relative to X_κ). Also, no point in $\mathbb{T}_\kappa^\omega$ is isolated (relative to X_κ). Thus, X_κ is crowded.

(4) Let $n \in \omega$ be such that $n \geq 1$. Being an open subspace of a compact Hausdorff space, $E \setminus \beta(2^\kappa)$ is locally compact. Thus, it follows from [15, Thm. 7] that $E \setminus \beta(2^\kappa)$ is n-resolvable. □

Theorem 8.13. *Each finite rooted* S4.2*-frame is an interior image of* X_κ*.*

Proof. Let $\mathfrak{F} = (W, R)$ be a finite rooted S4.2-frame. Let $C \subseteq W$ be the unique maximal cluster of $\mathfrak{F}$ and $\mathfrak{C}$ be the subframe of $\mathfrak{F}$ whose underlying set is C. Suppose C has $n \geq 1$ elements. Either $W = C$ or not. Assume $W = C$. It follows from Lemma 8.12(4) and [4, Lem. 5.9] that $\mathfrak{C}$ is an interior image of the subspace $E \setminus \beta(2^\kappa)$, say via $g : E \setminus \beta(2^\kappa) \to C$. Then any $f : X_\kappa \to W$ that extends g is an interior mapping onto $\mathfrak{F}$.

Assume $W \neq C$ and let $\mathfrak{G}$ be the subframe of $\mathfrak{F}$ whose underlying set is $W \setminus C$. Then $\mathfrak{G}$ is a finite rooted S4-frame. As demonstrated in the proof of Theorem 8.4, $\mathfrak{G}$ is an interior image of $\mathbb{T}_\kappa^\omega$. Because $\mathbb{T}_\kappa^\omega$ is a closed nowhere dense subspace of X_κ such that $X_\kappa \setminus \mathbb{T}_\kappa^\omega = E \setminus \beta(2^\kappa)$ is n-resolvable, we may apply Lemma 8.3 to obtain that $\mathfrak{F}$ is an interior image of X_κ. □

Theorem 8.14. *The logic of* X_κ *is* S4.2*.*

Proof. By Lemma 8.12(3), X_κ is ED. Thus, $\mathsf{S4.2} \subseteq \mathsf{Log}(X_\kappa)$. For the converse, suppose $\mathsf{S4.2} \not\vdash \varphi$. Then there is a finite rooted S4.2-frame $\mathfrak{F}$ refuting φ. Theorem 8.13 yields that $\mathfrak{F}$ is an interior image of X_κ. Thus, X_κ also refutes φ, giving that $\mathsf{S4.2} = \mathsf{Log}(X_\kappa)$. □

Remark 8.15. We conclude by summarizing the logics obtained through the preceding tree based constructions; see Figure 1. In the setting of trees with countable branching, some well-known spaces and results are realized by utilizing the patch topology. In particular, the logics S4, S4.1, Grz, and Grz_n for $n \geq 1$ are realized via trees with countable branching, see Table 5.

Generalizing the patch topology to the σ-patch topology introduces interesting spaces in the setting of trees with uncountable branching. We again realize the same logics in this setting, but for non-metrizable zero-dimensional Hausdorff spaces, see Table 6. To obtain new topological completeness results for the logics S4.2, S4.1.2, Grz.2, and $\mathsf{Grz.2}_n$ with respect to Tychonoff ED-spaces, we embed trees with uncountable branching equipped with the σ-patch topology into appropriately chosen ED-spaces, see Table 7. All our proofs are performed within ZFC.

Logic	is the logic of
$\mathsf{Grz.2}_{n+2}$	$X^n_\kappa = \mathbb{T}^n_\kappa \cup 2^\kappa \ (n \in \omega)$
$\mathsf{Grz.2}$	$\bigoplus_{n\in\omega} X^n_\kappa$
$\mathsf{S4.1.2}$	$X^\omega_\kappa = \mathbb{T}^\omega_\kappa \cup 2^\kappa$
$\mathsf{S4.2}$	$X_\kappa = \mathbb{T}_\kappa \cup (E \setminus \beta(2^\kappa))$

TABLE 7. Logics arising from embeddings in the uncountable branching case.

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References

[1] M. A. Abashidze, *Algebraic analysis of the Gödel-Löb modal system*, Ph.D. thesis, Tbilisi State University, 1987, In Russian.

[2] M. Aiello, J. van Benthem, and G. Bezhanishvili, *Reasoning about space: The modal way*, J. Logic Comput. **13** (2003), no. 6, 889–920.

[3] J. van Benthem, G. Bezhanishvili, B. ten Cate, and D. Sarenac, *Multimodal logics of products of topologies*, Studia Logica **84** (2006), no. 3, 369–392.

[4] G. Bezhanishvili, N. Bezhanishvili, J. Lucero-Bryan, and J. van Mill, *Krull dimension in modal logic*, J. Symb. Logic **82** (2017), no. 4, 1356–1386.

[5] ———, *On modal logics arising from scattered locally compact Hausdorff spaces*, Ann. Pure Appl. Logic **170** (2019), no. 5, 558–577.

[6] G. Bezhanishvili, L. Esakia, and D. Gabelaia, *Some results on modal axiomatization and definability for topological spaces*, Studia Logica **81** (2005), no. 3, 325–355.

[7] G. Bezhanishvili, D. Gabelaia, and J. Lucero-Bryan, *Modal logics of metric spaces*, Rev. Symb. Log. **8** (2015), no. 1, 178–191.

[8] ———, *Topological completeness of logics above S4*, J. Symb. Logic **80** (2015), no. 2, 520–566.

[9] G. Bezhanishvili and M. Gehrke, *Completeness of S4 with respect to the real line: revisited*, Ann. Pure Appl. Logic **131** (2005), no. 1-3, 287–301.

[10] G. Bezhanishvili and J. Harding, *The modal logic of* $\beta(\mathbb{N})$, Arch. Math. Logic **48** (2009), no. 3-4, 231–242.

[11] ———, *Modal logics of Stone spaces*, Order **29** (2012), no. 2, 271–292.

[12] G. Bezhanishvili and P. J. Morandi, *Scattered and hereditarily irresolvable spaces in modal logic*, Arch. Math. Logic **49** (2010), no. 3, 343–365.

[13] P. Blackburn, M. de Rijke, and Y. Venema, *Modal logic*, Cambridge University Press, 2001.

[14] A. Blass, *Infinitary combinatorics and modal logic*, J. Symbolic Logic **55** (1990), no. 2, 761–778.

[15] J. G. Ceder, *On maximally resolvable spaces*, Fund. Math. **55** (1964), 87–93.

[16] A. Chagrov and M. Zakharyaschev, *Modal logic*, Oxford University Press, 1997.

[17] A. Dow and J. van Mill, *An extremally disconnected Dowker space*, Proc. Amer. Math. Soc. **86** (1982), no. 4, 669–672.

[18] R. Engelking, *General topology*, second ed., Heldermann Verlag, Berlin, 1989.

[19] L. Gillman and M. Jerison, *Rings of continuous functions*, Springer-Verlag, New York-Heidelberg, 1976, Reprint of the 1960 edition, Graduate Texts in Mathematics, No. 43.

[20] P. T. Johnstone, *Stone spaces*, Cambridge University Press, Cambridge, 1982.

[21] P. Kremer, *Strong completeness of S4 for any dense-in-itself metric space*, Rev. Symb. Log. **6** (2013), no. 3, 545–570.

[22] J. C. C. McKinsey and A. Tarski, *The algebra of topology*, Annals of Mathematics **45** (1944), 141–191.

[23] G. Mints, *A completeness proof for propositional S4 in Cantor space*, Logic at work, Stud. Fuzziness Soft Comput., vol. 24, Physica, Heidelberg, 1999, pp. 79–88.

[24] J. Picado and A. Pultr, *Frames and locales*, Frontiers in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2012.

[25] H. Rasiowa and R. Sikorski, *The mathematics of metamathematics*, Monografie Matematyczne, Tom 41, Państwowe Wydawnictwo Naukowe, Warsaw, 1963.

[26] D. A. Rose, α*-scattered spaces*, Internat. J. Math. Math. Sci. **21** (1998), no. 1, 41–46.

[27] Z. Semadeni, *Banach spaces of continuous functions. Vol. I*, PWN—Polish Scientific Publishers, Warsaw, 1971.

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